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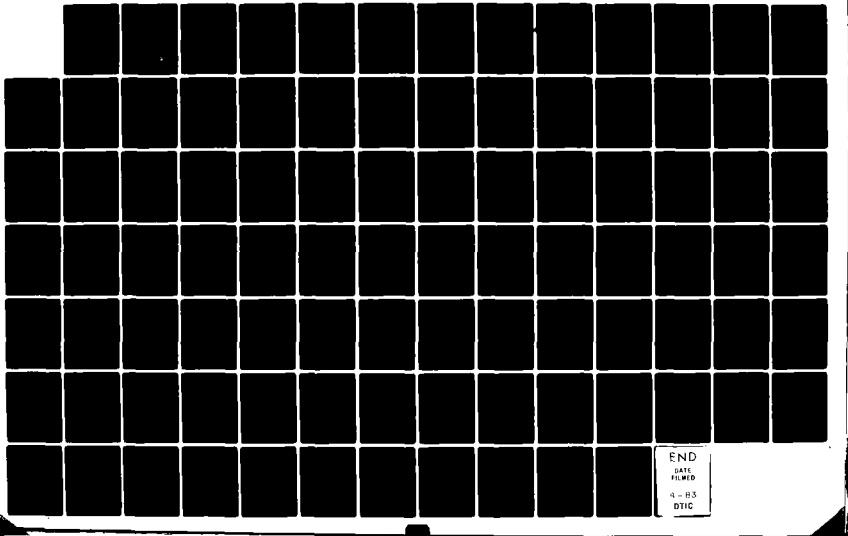

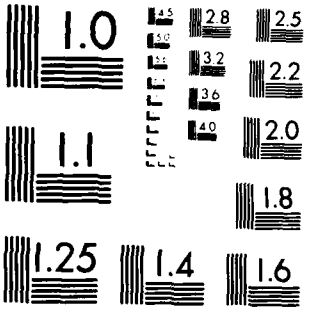

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ON THE PERMANENCE PROPERTY IN SPHERICAL SPLINE INTERPOLATION

WILLI FREEDEN

DEPARTMENT OF GEODETIC SCIENCE AND SURVEYING
THE OHIO STATE UNIVERSITY
COLUMBUS, OHIO 43210

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Spherical spline functions are introduced by use of Green's (surface) functions with respect to the Beltrami operator on the sphere. The method of interpolation by spherical splines is formulated as variational problem of minimizing a (Sobolev) "energy" norm under interpolatory constraints. The process is constructed so as to have the so - called permanence property, i.e. the transition from the interpolating spline with respect to N data to the interpolating spline with respect to N + 1 data necessitates merely the addition of one more term, all the terms obtained formerly remaining

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unchanged. The algorithm is numerically stable and very economical as regards the number of operations.

FOREWORD

This report was prepared by Dr. Willi Freeden, Associate Professor, Institute for Pure and Applied Mathematics, West Germany and Post Doctoral Research, Department of Geodetic Science and Surveying, The Ohio State University, under Air Force Contract No. F19628-82-0017, Project Supervisor Urho A. Uotila, Professor, Department of Geodetic Science and Surveying. The contract covering this research is administered by the Air Force Geophysics Laboratory, Hanscom Air Force Base, Massachusetts, with Dr. Christopher Jekeli, Contract Manager.

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1. INTRODUCTION

The basic problem of periodic interpolation can be explained as follows:

- given a finite set of distinct points $t_1, \ldots, t_N$ of the interval $[0, 2\pi)$ and a set of real scalars $y_1, \ldots, y_N$
- construct a 2π - periodic function s belonging to the linear variety V_N of all interpolants of a suitable space $\mathcal{H}$

$$V_N = \{ f \in \mathcal{H} \mid f(t_k) = y_k, k = 1, \ldots, N \}. \quad (1.1)$$

For practically relevant candidates of solution additional information is clearly needed, in particular polynomial precision and smoothness. This essentially amounts to restricting the set of $\mathcal{H}$ - interpolants by constraints involving a suitable "energy" norm.
In this connection a quadratic (semi-) norm generated by the integral

$$\left\{ \int_0^{2\pi} | D_m f(x) |^2 \, dx \right\}^{1/2} \quad (1.2)$$

has proved most efficient, where the differential operator

$$D_m = \left(\left(\frac{d}{dx}\right)^2 + 0^2\right)\left(\left(\frac{d}{dx}\right)^2 + 1^2\right) \ldots \left(\left(\frac{d}{dx}\right)^2 + m^2\right) \quad (1.3)$$

annihilates all trigonometric polynomials

$$S(x) = \sum_{k=0}^{m} \left(a_k \cos(kx) + b_k \sin(kx) \right) \quad (1.4)$$

of order m.

A proper setting for the problem of minimizing the (quadratic) functional (1.2) under interpolatory constraints is provided by the class $\mathcal{H} = \mathcal{H}^{(2m+2)}[0, 2\pi]$ of 2π - periodic functions, whose (distributional) derivatives up to the $(2m + 2)$ -th order are square-integrable on the interval $[0, 2\pi]$.

The solution s of the minimization problem

$$\int_0^{2\pi} |D_m s(x)|^2 dx = \inf_{f \in V_N} \int_0^{2\pi} |D_m f(x)|^2 dx \qquad (1.5)$$

is called (optimal) trigonometric spline interpolant in $\mathcal{H}^{(2m+2)}[0, 2\pi]$.

The trigonometric spline interpolant is essentially given by the following properties (cf. Schoenberg (1964)):

(i) s is 2π - periodic and continuous

(ii) s is infinitely differentiable for all points $t \in [0, 2\pi)$ $t \neq t_k$, $k = 1, \ldots, N$ with $D_m^2 s(x) = 0$, i.e.: s reduces to a trigonometric polynomial of order m for all points $t \in [0, 2\pi)$ with $t \neq t_k$, $k = 1, \ldots, N$.

(iii) $s(t_k) = y_k$ for $k = 1, \ldots, N$.

Spline interpolation turns out to be a most adaptable method to data for (global) interpolation, and the (semi-) norm (1.2) is a natural setting to maintain tne flexibility of piecewise trigonometric polynomials while at the same time achieving some degree of global smoothness. Of particular usefulness is the spline interpolant corresponding to the quadratic integral mean of the (linearized) curvature $(d/dx)^2 f$

$$\int_0^{2\pi} |D_0 f(x)|^2 \, dx \quad , \tag{1.6}$$

since the quantity (1.6) may be physically interpreted as the potential energy of a(statically deflected)thin beam (which indeed is proportional to the integral of the square of the (linearized) curvature of the elastica of the beam) (cf. Moritz (1978)).

Trigonometric splines may be interpreted as the spline functions for the (unit) circle, i.e. trigonometric splines are the two-dimensional analogues of the spherical splines discussed in this paper.

□

Roughly speaking, the spherical interpolation problem to be of importance for geodetically relevant purposes can be formulated as follows:

- given a finite set of distinct points (stations) $\zeta_1, \cdots, \zeta_N$ of the (unit) sphere Ω and a set $y_1, \ldots, y_N$ of real scalars (observations or measurements)
- find a (smooth) function $s : \Omega \longrightarrow \mathbb{R}$ belonging to the linear variety of all interpolants of a suitable space $\mathcal{H}$

$$V_N = \{ f \in \mathcal{H} \mid f(\zeta_k) = y_k \, , \, k = 1, \ldots, N \} . \tag{1.7}$$

In order to achieve uniqueness it likewise seems quite natural to look for an interpolant minimizing an appropriate quadratic norm in such a way that additional assumptions concerning polynomial precision and smoothness again are satisfied. This can be done in the same way as in the trigonometric case by using a differential operator annihilating all polynomials, i.e. spherical harmonics of order m or less. Observing the fact that the

spherical harmonics S_n of order n are the everywhere on the (unit) sphere regular eigenfunctions of the Beltrami operator Δ^* corresponding to the eigenvalues $\lambda_n = n(n+1)$, i.e.

$$(\Delta^*_\xi + \lambda_n)\, S_n(\xi) = 0 \tag{1.8}$$

on the unit sphere Ω for all spherical harmonics S_n of order n, the (simplest) operator playing the same role as the operator (1.3) is given by the product

$$(\Delta^*)_m = (\Delta^* + \lambda_0)(\Delta^* + \lambda_1) \ \ldots \ (\Delta^* + \lambda_m). \tag{1.9}$$

A proper setting for optimal spherical interpolation is provided by the (Sobolev) space $\mathcal{H} = \mathcal{H}^{(2m+2)}(\Omega)$ of functions whose (Beltrami) derivatives up to $(\Delta^*)_m f$ (in the distributional sense) are square-integrable on the sphere Ω (cf. Chapt. 8). Equipped with the (semi-) norm $\|\cdot\|_m = \sqrt{\langle \cdot, \cdot \rangle_m}$ generated by

$$\langle f, f \rangle_m = \int_\Omega |(\Delta^*_\xi)_m f(\xi)|^2 \, d\omega \tag{1.10}$$

the linear space $\mathcal{H}^{(2m+2)}(\Omega)$ is a (semi-) Hilbert function space of continuous functions on the (unit) sphere Ω. Therefore, in comparison with (1.5), the (optimal) spherical interpolation problem to be analyzed in this paper can be formulated as follows:

- find a function $s \in V_N$ such that

$$\int_\Omega |(\Delta^*_\xi)_m s(\xi)|^2 d\omega = \inf_{f \in V_N} \int_\Omega |(\Delta^*_\xi)_m f(\xi)|^2 d\omega . \tag{1.11}$$

As described in detail the variational problem (1.11) is well-posed in the sense that its solution exists, is unique, and depends continuously on the data $y_1, \ldots, y_N$. Basic tool is the theory of Green's functions of the sphere with respect to the operator (1.9).

The spherical spline interpolant is essentially given by the following properties:

(i) s is continuous on Ω

(ii) s is infinitely differentiable for all points $\eta \in \Omega$, $\eta \neq \eta_k$, $k = 1, \ldots, N$ with $(\Delta^* + \lambda_0)^2 \ldots (\Delta^* + \lambda_m)^2 s(\eta) = 0$, i.e. s reduces to a polynomial of order $\leq m$ for all points $\eta \in \Omega$ with $\eta \neq \eta_k$, $k = 1, \ldots, N$

(iii) $s(\eta_k) = y_k$ for $k = 1, \ldots, N$.

Spherical spline functions (s.s.f.) have the following attractive features:

The linear space of all s.s.f. (of order m) is finite dimensional; s.s.f. are relatively easy to manipulate and compute; various matrices arising in interpolation and approximation problems are nonsingular, s.s.f. do not tend to produce approximations having severe oscillations and undulations.

Thus s.s.f. seem to be best suited for macro - and micro modeling of earth's gravitational field. Moreover, a variety of problems of "optimal" integration on the sphere leads to spherical spline functions. In addition, the whole solution process can be made surprisingly simple and efficient for numerical computation. Indeed, the computational scheme for solving the interpolation problem can be described in a recursive form based on a combination of generalized Lagrange and Newton formula. This means that the method is constructed so as to have the so - called

permanence property: the transition from the solution with respect to N data to the solution with respect to (N + 1) - data necessitates merely the addition of one more term, all the terms obtained formerly remaining unchanged.

The price to be paid for the convenience of the permanence property is a biorthonormalization process.

□

2. DEFINITIONS AND NOTATIONS

$\mathbb{R}^3$ denotes the three-dimensional real Euclidean space. We consistently write $x, y, z, \ldots$ for the elements of $\mathbb{R}^3$.

Let e_1, e_2, e_3 be the canonical (orthonormal) basis in $\mathbb{R}^3$

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{2.1}$$

In components we have, for elements $x, z \in \mathbb{R}^3$,

$$\begin{aligned} x &= x_1 e_1 + x_2 e_2 + x_3 e_3 \\ z &= z_1 e_1 + z_2 e_2 + z_3 e_3 . \end{aligned} \tag{2.2}$$

The inner product of $x, z \in \mathbb{R}^3$ is the number

$$x \cdot z = x_1 z_1 + x_2 z_2 + x_3 z_3 , \tag{2.3}$$

the norm of $x \in \mathbb{R}^3$ is the (nonnegative) number

$$|x| = \sqrt{x \cdot x} . \tag{2.4}$$

For all elements $x \in \mathbb{R}^3$, different from the origin, we have the representation

$$x = r \xi , \quad r = |x| = \sqrt{x_1^2 + x_2^2 + x_3^2} , \tag{2.5}$$

where ξ is the uniquely determined directional unit vector of the element $x \in \mathbb{R}^3$.

The unit sphere in $\mathbb{R}^3$ will be called Ω. The total surface of Ω will be denoted by ω :

$$\omega = \int_{\Omega} d\omega = 4\pi . \qquad (d\omega : \text{surface element}).$$

The rectangular coordinates (2.1) are related to the polar coordinates (2.5) by the equations

$$x = r\,\xi \;, \quad \xi = t e_3 + \sqrt{1-t^2}\,(\cos\varphi\, e_1 + \sin\varphi\, e_2) \qquad (2.6)$$
$$t = \cos\vartheta \;, \quad 0 \le \vartheta \le \pi \;, \quad 0 \le \varphi < 2\pi,$$

i.e.:

$$\begin{aligned} x_1 &= r\sqrt{1-t^2}\,\cos\varphi \\ x_2 &= r\sqrt{1-t^2}\,\sin\varphi \\ x_3 &= r\,t \end{aligned} \qquad (2.7)$$

(ϑ: polar distance, φ: geocentric longitude). In terms of the coordinates (2.6) the Laplace - operator

$$\Delta_x = \left(\frac{\partial}{\partial x_1}\right)^2 + \left(\frac{\partial}{\partial x_2}\right)^2 + \left(\frac{\partial}{\partial x_3}\right)^2 \qquad (2.8)$$

takes the form

$$\Delta_x = \left(\frac{\partial}{\partial r}\right)^2 + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\,\Delta^*_\xi . \qquad (2.9)$$

Δ^* denotes the (Laplace -) Beltrami - operator of the unit sphere

$$\Delta^*_\xi = (1-t^2)\left(\frac{\partial}{\partial t}\right)^2 - 2t\frac{\partial}{\partial t} + \frac{1}{1-t^2}\left(\frac{\partial}{\partial\varphi}\right)^2 \qquad (2.10)$$

$$= \frac{\partial}{\partial t}(1-t^2)\frac{\partial}{\partial t} + \frac{1}{1-t^2}\left(\frac{\partial}{\partial\varphi}\right)^2 . \qquad \square$$

3. SPHERICAL HARMONICS

The (Laplace -) spherical harmonics S_n of order n are defined as the everywhere on the unit sphere Ω infinitely differentiable eigenfunctions of the Beltrami differential equation

$$(\Delta^*_\xi + \lambda_n)\, S_n(\xi) = 0 \tag{3.1}$$

corresponding to the eigenvalues

$$\lambda_n = n(n+1) . \tag{3.2}$$

The set of all eigenvalues is the spectrum

$$S(\Omega) = \{ \lambda_n = n(n+1) \mid n = 0, 1, \ldots \} . \tag{3.3}$$

As is well-known, the functions H_n given by

$$H_n(x) = r^n\, S_n(\xi) \quad , \quad r = |x| \quad , \quad \xi \in \Omega, \tag{3.4}$$

are polynomials in rectangular coordinates which satisfy the Laplace equation

$$\Delta_x\, H_n(x) = 0 \tag{3.5}$$

and are homogeneous of degree n. Conversely, every homogeneous harmonic polynomial of degree n restricted to the unit sphere Ω is a spherical harmonic of order n.

The Legendre polynomials P_n given by

$$P_n(t) = \sum_{s=0}^{[n/2]} (-1)^s \frac{(2n-2s)!}{2^n (n-2s)!\,(n-s)!\,s!}\, t^{n-2s} \tag{3.6}$$

are the only everywhere on the interval $[-1, 1]$ infinitely differentiable eigenfunctions of the Legendre differential equation

$$\left[\frac{d}{dt} (1 - t^2) \frac{d}{dt} + \lambda_n \right] P_n(t) = 0 \quad , \quad t \in [-1, 1] , \tag{3.7}$$

which in $t = 1$ satisfy $P_n(1) = 1$.

Apart from a constant factor, the Legendre polynomials are the only spherical harmonics, which are invariant under orthogonal transformations with the "north-pole" e_3 as fixed point.

□

Spherical harmonics of different order are orthogonal in the sense of the $\mathcal{L}^2$ - inner product

$$(S_n , S_m) = \int_{\Omega} S_n(\xi) S_m(\xi) \, d\omega = 0 . \tag{3.8}$$

$$(n \neq m)$$

The linear space $\mathcal{S}_n$ of all spherical harmonics of order n has the dimension

$$dim(\mathcal{S}_n) = 2n + 1 . \tag{3.9}$$

In other words, there must be $2n + 1$ linearly independent spherical harmonics

$$S_{n,1} , \ldots , S_{n,2n+1} . \tag{3.10}$$

We assume this system to be orthonormalized in the sense of the $\mathcal{L}^2$ -inner product

$$(S_{n,j}, S_{m,k}) = \int_{\Omega} S_{n,j}(\xi)\, S_{m,k}(\xi)\, d\omega \qquad (3.11)$$

$$= \delta_{nm}\, \delta_{jk}.$$

(δ_{nm}: Kronecker symbol) □

$\mathcal{P}_m$ denotes the linear space of all functions in three variables which restricted to the unit sphere Ω may be represented by homogeneous polynomials of degree m or less. $\mathcal{P}_m$ admits the orthogonal decomposition

$$\mathcal{P}_m = \mathcal{S}_0 \oplus \ldots \oplus \mathcal{S}_m$$

with respect to the $\mathcal{L}_2$-inner product, i.e. to every $S \in \mathcal{P}_m$ there exist spherical harmonics $S_0, \ldots, S_m$ with

$$S = \sum_{n=0}^{m} S_n \quad , \quad (S_n, S_j) = 0 \quad . \qquad (3.12)$$

$$(n \neq j \; , \; 0 \leq j, n \leq m)$$

Hence, the dimension of $\mathcal{P}_m$ is equal to

$$M = \dim(\mathcal{P}_m) = \sum_{n=0}^{m} (2n+1) = (m+1)^2. \qquad (3.13)$$

□

For any two elements $\xi, \eta \in \Omega$, the sum

$$F_n(\xi, \eta) = \sum_{j=1}^{2n+1} S_{n,j}(\xi)\, S_{n,j}(\eta)$$

is invariant under all orthogonal transformations A, i.e.:

$$F_n(\xi, \eta) = F_n(A\xi, A\eta) \qquad (3.14)$$

for all orthogonal transformations A. For fixed $\xi \in \Omega$, $\bar{r}_n(\xi, \eta)$ is as function of η a spherical harmonic of order n. $\bar{r}_n(\xi, \eta)$ is symmetric in ξ, η and depends only on the scalar product of ξ and η. Thus it is clear from the above that we have, apart from a multiplicative constant α_n,

$$\sum_{j=1}^{2n+1} S_{n,j}(\xi)\, S_{n,j}(\eta) = \alpha_n P_n(\xi \cdot \eta). \tag{3.15}$$

In order to determine α_n we set $\xi = \eta$. Then we obtain

$$\sum_{j=1}^{2n+1} |S_{n,j}(\xi)|^2 = \alpha_n P_n(1) = \alpha_n. \tag{3.16}$$

Integration over Ω yields

$$2n+1 = 4\pi\, \alpha_n. \tag{3.17}$$

Therefore we find the addition theorem

$$\sum_{j=1}^{2n+1} S_{n,j}(\xi)\, S_{n,j}(\eta) = \frac{2n+1}{4\pi} P_n(\xi \cdot \eta). \tag{3.18}$$

In particular, we have

$$\sum_{n=0}^{m} \sum_{j=1}^{2n+1} S_{n,j}(\xi)\, S_{n,j}(\eta) = \sum_{n=0}^{m} \frac{2n+1}{4\pi} P_n(\xi \cdot \eta). \tag{3.19}$$

Let Φ be an absolutely integrable function on the interval $[-1,1]$, i.e: $\Phi \in L^1[-1,1]$. Then, for every $S_n \in \mathcal{S}_n$, Hecke's formula (cf. Müller (1966), Freeden (1979)) gives

$$\int_{\Omega} \Phi(\xi \cdot \eta)\, S_n(\eta)\, d\omega(\eta) = \mu_n S_n(\xi), \tag{3.20}$$

where

$$\mu_n = 2\pi \int_{-1}^{+1} \phi(t)\ P_n(t)\ dt. \tag{3.21}$$

The notation $d\omega(\gamma)$ means the surface element $d\omega$ is applied to the γ variable.

Hecke's formula establishes the close connection between the orthogonal invariance of the sphere and the addition theorem.

□

4. FUNDAMENTAL SYSTEMS

Unfortunately, the system of spherical harmonics

$$Y_{0,1}, \ldots, Y_{n,1}, \ldots, Y_{n,2n+1}, \ldots, Y_{m,1}, \ldots, Y_{m,2m+1} \tag{4.1}$$

is not unisolvent on the unit sphere Ω, i.e. the matrix

$$\begin{pmatrix} Y_{0,1}(\eta_1) & \cdots & Y_{0,1}(\eta_M) \\ Y_{1,1}(\eta_1) & \cdots & Y_{1,1}(\eta_M) \\ \vdots & & \vdots \\ Y_{1,3}(\eta_1) & \cdots & Y_{1,3}(\eta_M) \\ \vdots & & \vdots \\ Y_{m,1}(\eta_1) & \cdots & Y_{m,1}(\eta_M) \\ \vdots & & \vdots \\ Y_{m,2m+1}(\eta_1) & \cdots & Y_{m,2m+1}(\eta_M) \end{pmatrix} \tag{4.2}$$

is not non-degenerate for all choices of M distinct points $\eta_1, \ldots, \eta_M$ lying on Ω. This is known from Haar's theorem (cf. Davis (1963) Theorem 2.41)). However, it is easy to prove that there exist systems of points $\eta_1, \ldots, \eta_M$ having a non-degenerate matrix (4.2). For, it is certainly possible to find a point η_1 with $Y_{0,1}(\eta_1) \neq 0$. We consider the determinant

$$\det \begin{pmatrix} Y_{0,1}(\eta_1) & Y_{0,1}(\xi) \\ Y_{1,1}(\eta_1) & Y_{1,1}(\xi) \end{pmatrix} . \tag{4.3}$$

As a function of ξ, this determinant cannot be identically 0, for else $Y_{0,1}$ and $Y_{1,1}$ would be linearly dependent. Therefore there is a

point η_2 such that

$$\det \begin{pmatrix} S_{0,1}(\eta_1) & S_{0,1}(\eta_2) \\ S_{1,1}(\eta_1) & S_{1,1}(\eta_2) \end{pmatrix} \neq 0. \qquad (4.4)$$

In the same way we discuss the determinant

$$\det \begin{pmatrix} S_{0,1}(\eta_1) & S_{0,1}(\eta_2) & S_{0,1}(\xi) \\ S_{1,1}(\eta_1) & S_{1,1}(\eta_2) & S_{1,1}(\xi) \\ S_{1,2}(\eta_1) & S_{1,2}(\eta_2) & S_{1,2}(\xi) \end{pmatrix} \qquad (4.5)$$

and by the same arguments there exists a point $\xi = \eta_3$ for which (4.5) is different from zero. Therefore, by induction, we can find a system of points $\eta_1, \dots, \eta_M$ such that the matrix (4.2) is non-degenerate.

A set $\eta_1, \dots, \eta_M$ of $M = (m+1)^2$ points of the unit sphere Ω is called <u>fundamental system</u> of order m on Ω, if the rank of the (M,M) - matrix

$$\alpha^M = \begin{pmatrix} S_{0,1}(\eta_1) & \cdots & S_{0,1}(\eta_M) \\ S_{1,1}(\eta_1) & \cdots & S_{1,1}(\eta_M) \\ \vdots & & \vdots \\ S_{1,3}(\eta_1) & \cdots & S_{1,3}(\eta_M) \\ \vdots & & \vdots \\ S_{m,1}(\eta_1) & \cdots & S_{m,1}(\eta_M) \\ \vdots & & \vdots \\ S_{m,2m+1}(\eta_1) & \cdots & S_{m,2m+1}(\eta_M) \end{pmatrix} \qquad (4.6)$$

is equal to M.

A set $\eta_1, \ldots, \eta_N$ of $N \geq M = (m+1)^2$ (distinct) points of the unit sphere Ω is called admissible system of order m on Ω, if the rank of the (M,N) - matrix

$$\alpha^N = \begin{pmatrix} S_{0,1}(\eta_1) & \cdots & S_{0,1}(\eta_N) \\ S_{1,1}(\eta_1) & \cdots & S_{1,1}(\eta_N) \\ \vdots & & \vdots \\ S_{1,3}(\eta_1) & \cdots & S_{1,3}(\eta_N) \\ \vdots & & \vdots \\ S_{m,1}(\eta_1) & \cdots & S_{m,1}(\eta_N) \\ \vdots & & \vdots \\ S_{m,2m+1}(\eta_1) & \cdots & S_{m,2m+1}(\eta_N) \end{pmatrix} \tag{4.7}$$

is equal to M.

NOTE: In the sequel we assume that each admissible system $\eta_1, \ldots, \eta_N$ of order m has as subset the fundamental system $\eta_1, \ldots, \eta_M$ of order m (consisting of the first M elements $\eta_1, \ldots, \eta_M$). This is always achievable by reordering.

Given a function $S \in \mathcal{P}_m$ of the (general) form

$$S(\xi) = \sum_{n=0}^{m} \sum_{j=1}^{2n+1} c_{n,j} \, S_{n,j}(\xi) . \tag{4.8}$$

For an admissible system $\eta_1, \ldots, \eta_N$ of order m on Ω, the linear equations

$$\sum_{k=1}^{N} a_k \, S_{n,j}(\eta_k) = c_{n,j} \qquad (n = 0, \ldots, m; \; j = 1, \ldots, 2n+1) \tag{4.9}$$

are solvable in the unknowns $a_1, \ldots, a_N$. Thus, for every admissible system $\eta_1, \ldots, \eta_N$ of order m on Ω and all solutions $a_1, \ldots, a_N$ of the linear equations (4.9) the element $S \in \mathcal{P}_m$ of the form (4.8) can be expressed as follows:

$$S(\xi) = \sum_{k=1}^{N} a_k \sum_{n=0}^{m} \sum_{j=1}^{2n+1} S_{n,j}(\eta_k)\, S_{n,j}(\xi). \tag{4.10}$$

Using the addition theorem we obtain

$$S(\xi) = \sum_{k=1}^{N} a_k \sum_{n=0}^{m} \frac{2n+1}{4\pi} P_n(\xi \cdot \eta_k). \tag{4.11}$$

But this means that the N functions given by

$$\begin{aligned} &\sum_{n=0}^{m} \frac{2n+1}{4\pi} P_n(\xi \cdot \eta_1) \\ &\qquad \vdots \\ &\sum_{n=0}^{m} \frac{2n+1}{4\pi} P_n(\xi \cdot \eta_N) \end{aligned} \tag{4.12}$$

span the space $\mathcal{P}_m$:

$$\mathcal{P}_m = \operatorname{span}\left(\left(\sum_{n=0}^{m} \frac{2n+1}{4\pi} P_n(\xi \cdot \eta_k) \right)_{k=1,\ldots,N} \right). \tag{4.13}$$

□

5. GREEN'S FUNCTIONS

We next define the Green function of the unit sphere Ω with respect to an operator $\Delta^* + \lambda_n$, $\lambda_n \in S(\Omega)$ (cf. Freeden (1978/1979)). This function will be of great importance for the definition and the application of spherical spline functions.

We begin our considerations by introducing the definition:

A function $g(\lambda_n; \xi, \eta)$ is called Green's function of the unit sphere Ω with respect to the operator $\Delta^* + \lambda_n$ and the parameter $\xi \in \Omega$ if it satisfies the following properties:

(i) $g(\lambda_n; \xi, \eta)$ is as function of η, for fixed ξ, infinitely differentiable for all $\eta \in \Omega$ with $\eta \neq \xi$ (differentiability).

(ii) For all $\eta \in \Omega$ with $\eta \neq \xi$

$$(\Delta^*_\eta + \lambda_n)\, g(\lambda_n; \xi, \eta) = (2n+1)\, P_n(\xi \cdot \eta)$$

(differential equation).

(iii) For all $\eta \in \Omega$,

$$g(\lambda_n; \xi, \eta) + \log(1 - \xi \cdot \eta)$$

is continuously differentiable for all $\eta \in \Omega$ (characteristic singularity).

(iv) For all orthogonal transformations A

$$g(\lambda_n; A\xi, A\eta) = g(\lambda_n, \xi, \eta)$$

(rotational symmetry).

(v)
$$\int_\Omega g(\lambda_n; \xi, \eta)\, P_n(\xi \cdot \eta)\, d\omega(\eta) = 0$$

uniformly with respect to all $\xi \in \Omega$

(normalization).

We first prove the uniqueness of Green's function $\mathcal{G}(\lambda_n; \xi, \eta)$: Denote by $\mathcal{D}(\lambda_n; \xi, \eta)$ the difference of two Green's functions satisfying the properties (i) - (v). Then we have

(i)' $\mathcal{D}(\lambda_n; \xi, \eta)$ is as function of η, for fixed $\xi \in \Omega$, infinitely differentiable for all $\eta \in \Omega$ with $\eta \neq \xi$.

(ii)'For all $\eta \in \Omega$ with $\eta \neq \xi$

$$(\Delta^*_\eta + \lambda_n)\, \mathcal{D}(\lambda_n; \xi, \eta) = 0 .$$

(iii)'For all $\eta \in \Omega$, $\mathcal{D}(\lambda_n; \xi, \eta)$ is continuously differentiable.

(iv)'For all transformations A

$$\mathcal{D}(\lambda_n; A\xi, A\eta) = \mathcal{D}(\lambda_n; \xi, \eta)$$

(v)' $$\int_\Omega \mathcal{D}(\lambda_n; \xi, \eta)\, P_n(\xi \cdot \eta)\, d\omega(\eta) = 0$$

uniformly with respect to all $\xi \in \Omega$.

The properties (i)' - (iii)' show that $\mathcal{D}(\lambda_n; \xi, \eta)$ is an everywhere on the unit sphere Ω infinitely differentiable function satisfying the differential equation (ii)'. Therefore $\mathcal{D}(\lambda_n; \xi, \eta)$ must be a spherical harmonic of order n. Because of the property (iv)', $\mathcal{D}(\lambda_n; \xi, \eta)$ depends only on the scalar product $\xi \cdot \eta$. Consequently, we have

$$\mathcal{D}(\lambda_n; \xi, \eta) = \gamma_n P_n(\xi \cdot \eta).$$

From (v)' we deduce that $\gamma_n = 0$. But this means that Green's function $\mathcal{G}(\lambda_n; \xi, \eta)$ of the unit sphere Ω with respect to the operator $\Delta^* + \lambda_n$ and the parameter $\xi \in \Omega$ is uniquely determined by the defining properties (i) - (v).

Following Hilbert's approach to the theory of Green's functions (cf. Hilbert (1912)) we prove the existence of $\mathcal{G}(\lambda_n; \xi, \eta)$ by first giving an explicit representation of Green's function to the operator Δ^*.

An easy calculation shows that

$$-\log(1-\xi\cdot\eta) - 1 + \log 2$$

satisfies all the defining properties (i) - (v) of Green's function with respect to Δ^*. Hence, by virtue of the uniqueness, we have

$$\mathcal{G}(0;\xi,\eta) = -\log(1-\xi\cdot\eta) - 1 + \log 2. \tag{5.1}$$

In order to assure the existence of $\mathcal{G}(\lambda_n;\xi,\eta)$, $\lambda_n \neq 0$, we consider the integral equation

$$\begin{aligned}\mathcal{G}(\lambda_n;\xi,\eta) &= \mathcal{G}(0;\xi,\eta) \\ &+ \frac{\lambda_n}{4\pi}\int_{\Omega}\mathcal{G}(\lambda_n;\xi,\zeta)\,\mathcal{G}(0;\eta,\zeta)\,d\omega(\zeta) \\ &- \frac{1}{\lambda_n} - \frac{2n+1}{\lambda_n}P_n(\xi\cdot\eta),\end{aligned} \tag{5.2}$$

which establishes the close relation between Green's function $\mathcal{G}(\lambda_n;\xi,\eta)$ and the resolvent of the kernel $\mathcal{G}(0;\xi,\eta)$. It is not difficult to see that

$$\begin{aligned}&\int_{\Omega}\mathcal{G}(0;\xi,\eta)\,S_n(\eta)\,d\omega(\eta) \\ &= \int_{\Omega}\left[\frac{1}{\lambda_n} + \frac{2n+1}{\lambda_n}P_n(\xi\cdot\eta)\right]S_n(\eta)\,d\omega(\eta)\end{aligned} \tag{5.3}$$

for all spherical harmonics $S_n \in \mathcal{S}_n$ of order $n > 0$. The integral equation (5.2) therefore has a solution which is uniquely determined by the conditions

$$\int_{\Omega}\mathcal{G}(\lambda_n;\xi,\eta)\,S_n(\eta)\,d\omega(\eta) = 0 \tag{5.4}$$

for all $S_n \in \mathcal{S}_n$.

□

Observing the characteristic singularity of Green's function we obtain applying Green's (surface) identity

$$\frac{\lambda_k - \lambda_n}{4\pi} \int_{\Omega} \mathcal{G}(\lambda_n; \xi, \eta)\, S_n(\eta)\, d\omega(\eta) = (1 - \delta_{\lambda_k \lambda_n})\, S_n(\eta). \quad (5.5)$$

Thus the spherical harmonics of order n, i.e. the eigenfunctions S_n of the Beltrami - operator Δ^* with respect to the eigenvalues $\lambda_n = n(n+1)$ are eigenfunctions of Green's (kernel) function $\mathcal{G}(\lambda_n; \xi, \eta)$ in the sense of the integral equation (5.5). Furthermore, if $\xi, \eta \in \Omega$ with $-1 \leq \xi \cdot \eta < 1$, the kernel $\mathcal{G}(\lambda_n; \xi, \eta)$ allows the bilinear expansion

$$\mathcal{G}(\lambda_n; \xi, \eta) = \sum_{k=0}^{\infty}{}^{*} \frac{4\pi}{\lambda_k - \lambda_n} \sum_{j=1}^{2k+1} S_{k,j}(\xi)\, S_{k,j}(\eta). \quad (5.6)$$

The symbol $\sum^*$ denotes that the sum is to be extended over all nonnegative integers k with $\lambda_k \neq \lambda_n$.

□

Using the addition theorem we can rewrite the bilinear expansion of $\mathcal{G}(\lambda_n; \xi, \eta)$ in the form

$$\mathcal{G}(\lambda_n; \xi, \eta) = \sum_{k=0}^{\infty}{}^{*} \frac{2k+1}{\lambda_k - \lambda_n} P_k(\xi \cdot \eta). \quad (5.7)$$

According to the classical Fredholm - Hilbert theory of linear integral equations (cf. Hilbert (1912)) we define iterated Green's functions by the following convolutions

$$\mathcal{G}(\lambda_0, \ldots, \lambda_m; \xi, \eta) \quad (5.8)$$

$$= \int_{\Omega} \mathcal{G}(\lambda_0, \ldots, \lambda_{m-1}; \xi, \zeta)\, \mathcal{G}(\lambda_m; \zeta, \eta)\, d\omega(\zeta)$$

$$(m = 1, 2, \ldots)$$

$g(\lambda_0, \dots, \lambda_m; \xi, \eta)$ is called Green's function of the unit sphere with respect to the operator $(\Delta^*)_m = (\Delta^* + \lambda_0) \dots (\Delta^* + \lambda_m)$.

In analogy to techniques known in potential theory it can be shown that, for integers $m \geq 1$, $g(\lambda_0, \dots, \lambda_m; \xi, \eta)$ is continuous on the whole sphere Ω as function of η with ξ fixed or as function of ξ with η fixed.

On the other hand, the bilinear expansion of $g(\lambda_0, \dots, \lambda_m; \xi, \eta)$, $m \geq 1$,

$$(4\pi)^m \sum_{k=m+1}^{\infty} \frac{2k+1}{(\lambda_k - \lambda_0) \dots (\lambda_k - \lambda_m)} P_k(\xi \cdot \eta) \tag{5.9}$$

is absolutely and uniformly convergent both in ξ and η respectively and uniformly in ξ and η together. Thus the representation theorem of (generalized) Fourier theory yields

$$g(\lambda_0, \dots, \lambda_m; \xi, \eta) = (4\pi)^m \sum_{k=m+1}^{\infty} \frac{2k+1}{(\lambda_k - \lambda_0) \dots (\lambda_k - \lambda_m)} P_k(\xi \cdot \eta). \tag{5.10}$$

Let m be an integer with $m \geq 2$. Then the derivative

$$(\Delta^* + \lambda_0) \dots (\Delta^* + \lambda_{m-2})\, g(\lambda_0, \dots, \lambda_m; \xi, \eta) \tag{5.11}$$

is as function of η for fixed ξ, a continuous function on the unit sphere. For integers $m \geq 1$, the derivative

$$(\Delta^* + \lambda_0) \dots (\Delta^* + \lambda_{m-1})\, g(\lambda_0, \dots, \lambda_m; \xi, \eta) \tag{5.12}$$

possesses as function of η logarithmic singularity in $\xi \in \Omega$. Furthermore, for elements $\eta \in \Omega$ with $\eta \neq \xi$ and integers $m \geq 1$,

$$[(\Delta^* + \lambda_1) \dots (\Delta^* + \lambda_m)] \; \mathcal{G}(\lambda_0, \dots, \lambda_m; \xi, \eta)$$

$$= (-4\pi)^m \left[\mathcal{G}(\lambda_0; \xi, \eta) - \sum_{k=1}^{m} \frac{2k+1}{\lambda_k - \lambda_0} P_k(\xi \cdot \eta) \right]. \tag{5.13}$$

□

6. INTEGRAL FORMULAS

Let ξ be a fixed point of the unit sphere Ω. If now f is a function with continuous second derivatives on Ω, then for each sufficiently small $\rho > 0$ Green's surface identity gives

$$\int_{\substack{|\xi-\eta|\geq\rho \\ |\eta|=1}} \left\{ G(\lambda_0;\xi,\eta)\left[\Delta_\eta^* f(\eta)\right] - f(\eta)\left[\Delta_\eta^* G(\lambda_0;\xi,\eta)\right]\right\} d\omega(\eta) \qquad (6.1)$$

$$= \int_{\substack{|\xi-\eta|=\rho \\ |\eta|=1}} \left\{ G(\lambda_0;\xi,\eta)\left[\frac{\partial}{\partial n_\eta} f(\eta)\right] - f(\eta)\left[\frac{\partial}{\partial n_\eta} G(\lambda_0;\xi,\eta)\right]\right\} ds(\eta).$$

Herein ds is the line element in $\mathbb{R}^3$, while n is the unit vector, normal to the curve $|\xi-\eta| = \rho$ on Ω, tangential to Ω and directed exterior to the set of all $\eta \in \Omega$ with $|\xi - \eta| \geq \rho$.

In identity (6.1) we first observe the differential equation of Green's function

$$\int_{\substack{|\xi-\eta|\geq\rho \\ |\eta|=1}} f(\eta)\left[\Delta_\eta^* G(\lambda_0;\xi,\eta)\right] d\omega(\eta) = \int_{\substack{|\xi-\eta|\geq\rho \\ |\eta|=1}} f(\eta)\, d\omega(\eta). \qquad (6.2)$$

in polar coordinates (2.6) the line element for $|\xi - \eta| = \rho$ can be expressed by

$$ds^2(\eta) = \frac{1}{1-t^2}\, dt^2 + (1-t^2)\, d\phi^2.$$

Therefore, by virtue of the logarithmic singularity of Green's function $\mathcal{G}(\lambda_0; \xi, \zeta)$, we get in analogy to the well-known considerations of potential theory, on passing to the limit $\rho \to 0$ the theorem:

Let ξ be a fixed point of the unit sphere Ω. Suppose that f is a twice continuously differentiable function on Ω. Then

$$f(\xi) = \frac{1}{4\pi} \int_{\Omega} f(\zeta)\, d\omega(\zeta) \tag{6.3}$$

$$- \frac{1}{4\pi} \int_{\Omega} \mathcal{G}(0; \xi, \zeta) \left[\Delta_\zeta^* f(\zeta)\right] d\omega(\zeta).$$

This formula compares the functional value of a function $f \in C^{(2)}(\Omega)$ at $\xi \in \Omega$ with the mean (integral) value of f on the unit sphere Ω.

□

By use of Green's function $\mathcal{G}(\lambda_0, \lambda_1; \xi, \zeta)$ with respect to the operator $(\Delta^* + \lambda_0)(\Delta^* + \lambda_1)$ we are able to generalize the integral formula (6.3).

Observing the recursion property

$$(\Delta_\zeta^* + \lambda_1)\, \mathcal{G}(\lambda_0, \lambda_1; \xi, \zeta) \tag{6.4}$$

$$= (-4\pi) \left[\mathcal{G}(\lambda_0; \xi, \zeta) - \frac{3}{\lambda_1 - \lambda_0} P_1(\xi\zeta)\right]$$

we obtain

$$\int_{\Omega} G(\lambda_0; \xi, \eta)\,[(\Delta_\eta^* + \lambda_0) f(\eta)]\, d\omega(\eta) \tag{6.5}$$

$$= -\frac{1}{4\pi} \int_{\Omega} (\Delta^* + \lambda_1)\, G(\lambda_0, \lambda_1; \xi, \eta)\,[(\Delta_\eta^* + \lambda_0) f(\eta)]\, d\omega(\eta)$$

$$+ \int_{\Omega} \frac{3}{\lambda_1 - \lambda_0} P_1(\xi \cdot \eta)\,[(\Delta_\eta^* + \lambda_0) f(\eta)]\, d\omega(\eta).$$

Integration by parts, i.e. application of Green's identity yields for a function $f \in C^{(4)}(\Omega)$

$$-\frac{1}{4\pi} \int_{\Omega} (\Delta_\eta^* + \lambda_1)\, G(\lambda_0, \lambda_1; \xi, \eta)\,[(\Delta_\eta^* + \lambda_0) f(\eta)]\, d\omega(\eta)$$

$$= -\frac{1}{4\pi} \int_{\Omega} G(\lambda_0, \lambda_1; \xi, \eta)\,[(\Delta_\eta^* + \lambda_0)(\Delta_\eta^* + \lambda_1) f(\eta)]\, d\omega(\eta).$$

In the same way, by integration by parts, we get

$$\int_{\Omega} \frac{3}{\lambda_1 - \lambda_0} P_1(\xi \cdot \eta)\,[(\Delta_\eta^* + \lambda_0) f(\eta)]\, d\omega(\eta)$$

$$= -3 \int_{\Omega} P_1(\xi \cdot \eta)\, f(\eta)\, d\omega(\eta).$$

Therefore, we have

$$\int_{\Omega} \mathcal{G}(\lambda_0;\, \xi,\eta)\, [(\Delta_\eta^* + \lambda_0)\, f(\eta)]\, d\omega(\eta) \tag{6.6}$$

$$= -\frac{1}{4\pi} \int_{\Omega} \mathcal{G}(\lambda_0, \lambda_1;\, \xi,\eta)\, [(\Delta_\eta^* + \lambda_0)(\Delta_\eta^* + \lambda_1) f(\eta)]\, d\omega(\eta)$$

$$- 3 \int_{\Omega} f(\eta)\, P_1(\xi\cdot\eta)\, d\omega(\eta)$$

provided that f is a four-times continuously differentiable function on Ω. Thus, by combination of (6.3) and (6.6), we have for $f \in C^{(4)}(\Omega)$

$$f(\xi) = \sum_{n=0}^{1} \frac{2n+1}{4\pi} \int_{\Omega} f(\eta)\, P_n(\xi\cdot\eta)\, d\omega(\eta) \tag{6.7}$$

$$+ \left(-\frac{1}{4\pi}\right)^2 \int_{\Omega} \mathcal{G}(\lambda_0, \lambda_1;\, \xi,\eta)\, [(\Delta_\eta^* + \lambda_0)(\Delta_\eta^* + \lambda_1) f(\eta)]\, d\omega(\eta).$$

□

More generally, by successive integration by parts we obtain in connection with the definition of $\mathcal{G}(\lambda_0, \ldots, \lambda_m;\, \xi,\eta)$ and the formula (5.13) the integral formula:

Let ξ be a fixed point of the unit sphere Ω. Let f be a $(2m+2)$ - times continuously differentiable function on Ω. Then

$$f(\xi) = \sum_{n=0}^{m} \frac{2n+1}{4\pi} \int_{\Omega} f(\eta)\, P_n(\xi\cdot\eta)\, d\omega(\eta) \tag{6.8}$$

$$+ \left(-\frac{1}{4\pi}\right)^{m+1} \int_{\Omega} \mathcal{G}(\lambda_0, \ldots, \lambda_m;\, \xi,\eta)\, [(\Delta_\eta^* + \lambda_0) \ldots (\Delta_\eta^* + \lambda_m) f(\eta)]\, d\omega(\eta).$$

Inserting the addition theorem of spherical harmonics into the first term of the right hand side we find

$$f(\xi) = \sum_{n=0}^{m} \sum_{j=1}^{2n+1} S_{n,j}(\xi) \int_{\Omega} f(\eta)\, S_{n,j}(\eta)\, d\omega(\eta) \tag{6.9}$$

$$+ \left(-\frac{1}{4\pi}\right)^{m+1} \int_{\Omega} \mathcal{G}(\lambda_0, \dots, \lambda_m; \xi, \eta) \left[(\Delta_\eta^* + \lambda_0) \dots (\Delta_\eta^* + \lambda_m) f(\eta)\right] d\omega(\eta).$$

This formula gives a comparison between the m-th partial sum of the orthogonal expansion of f into spherical harmonics and the functional value of f taken at the point $\xi \in \Omega$ respectively (with explicit knowledge of the remainder term).

The formula (6.9) may be considered as the theoretical background for problems of interpolation and best approximation by spherical splines. In order to see this we have to modify our integral formula (6.9) using Green's function

$$\mathcal{G}^{(2)}(\lambda_0, \dots, \lambda_m; \xi, \eta) \tag{6.10}$$

$$= \int_{\Omega} \mathcal{G}(\lambda_0, \dots, \lambda_m; \xi, \zeta)\, \mathcal{G}(\lambda_0, \dots, \lambda_m; \zeta, \eta)\, d\omega(\zeta)$$

with respect to the operator $(\Delta^* + \lambda_0)^2 \dots (\Delta^* + \lambda_m)^2$ and the parameter $\xi \in \Omega$.

Inserting (5.10) into (6.10) we obtain as bilinear expansion

$$\mathcal{G}^{(2)}(\lambda_0, \dots, \lambda_m; \xi, \eta) = (4\pi)^{2m+1} \sum_{k=m+1}^{\infty} \frac{2k+1}{(\lambda_k - \lambda_0)^2 \dots (\lambda_k - \lambda_m)^2} P_k(\xi \cdot \eta). \tag{6.11}$$

Hence, it is obvious that

$$\mathcal{G}(\lambda_0, \dots, \lambda_m, \xi, \eta) \tag{6.12}$$

$$= \left(-\frac{1}{4\pi}\right)^{m+1} \left[(\Delta^* + \lambda_0) \dots (\Delta^* + \lambda_m)\, \mathcal{G}^{(2)}(\lambda_0, \dots \lambda_m; \xi, \eta)\right].$$

Summarizing our results we therefore obtain the theorem:

Let ξ be a fixed point of Ω. Suppose that f is a $(2m+2)$ - times continuously differentiable function defined on Ω. Then

$$f(\xi) \tag{6.13}$$

$$= \sum_{n=0}^{m} \sum_{j=1}^{2n+1} S_{n,j}(\xi) \int_{\Omega} f(\eta)\, S_{n,j}(\eta)\, d\omega(\eta)$$

$$+ \left(\frac{1}{4\pi}\right)^{2m+2} \int_{\Omega} \left[(\Delta_\eta^m + \lambda_0)\cdots(\Delta_\eta^m + \lambda_m) g^{(2)}(\lambda_0, \cdots, \lambda_m; \xi, \eta)\right]\left[(\Delta_\eta^m + \lambda_0)\cdots(\Delta_\eta^m + \lambda_m) f(\eta)\right] d\omega(\eta).$$

□

7. THE DIFFERENTIAL EQUATION $(\Delta^*)_m f = (\Delta^* + \lambda_0) \dots (\Delta^* + \lambda_m) f = g$.

Let S be an element of class $\mathcal{P}_m$ of the general form

$$S(\eta) = \sum_{n=0}^{m} \sum_{j=1}^{2n+1} c_{n,j} \, S_{n,j}(\eta). \tag{7.1}$$

Then, observing the differential equation

$$(\Delta_\eta^* + \lambda_n) \, S_{n,j}(\eta) = 0 \tag{7.2}$$

for $n = 0, \dots, m$ and for all $\eta \in \Omega$ we have

$$(\Delta_\eta^*)_m \, S(\eta) = (\Delta_\eta^* + \lambda_0) \dots (\Delta_\eta^* + \lambda_m) \, S(\eta) \tag{7.3}$$

$$= \sum_{n=0}^{m} \sum_{j=1}^{2n+1} c_{n,j} \, (\Delta_\eta^* + \lambda_0) \dots (\Delta_\eta^* + \lambda_m) \, S_{n,j}(\eta)$$

$$= 0$$

for all $\eta \in \Omega$.

On the other hand, it can be deduced from (6.13) that any solution $S \in C^{(2m+2)}(\Omega)$ of the homogeneous differential equation

$$(\Delta_\eta^* + \lambda_0) \dots (\Delta_\eta^* + \lambda_m) \, S(\eta) = 0 \tag{7.4}$$

is representable in the form

$$S(\eta) = \sum_{n=0}^{m} \sum_{j=1}^{2n+1} c_{n,j} \, S_{n,j}(\eta) \tag{7.5}$$

with coefficients $c_{n,j}$ given by

$$c_{n,j} = \int_{\Omega} S(\zeta)\, S_{n,j}(\zeta)\, d\omega(\zeta). \tag{7.6}$$

But this means that $\mathcal{P}_m$ is the null space of the operator

$$(\Delta^*)_m = (\Delta^* + \lambda_0) \ \ldots \ (\Delta^* + \lambda_m). \tag{7.7}$$

□

The integral formula (6.13) will be used now to discuss the general differential problem

$$(\Delta_\zeta^* + \lambda_0) \ \ldots \ (\Delta_\zeta^* + \lambda_m)\, f(\zeta) = g(\zeta) \tag{7.8}$$

for $\zeta \in \Omega$ and $g \in C(\Omega)$.
From Green's surface identity it follows that

$$\int_{\Omega} \left[(\Delta_\zeta^* + \lambda_0) \ \ldots \ (\Delta_\zeta^* + \lambda_m)\, f(\zeta)\right] S(\zeta)\, d\omega \tag{7.9}$$

$$- \int_{\Omega} \left[(\Delta_\zeta^* + \lambda_0) \ \ldots \ (\Delta_\zeta^* + \lambda_m)\, S(\zeta)\right] f(\zeta)\, d\omega$$

$$= 0$$

for all elements S belonging to $\mathcal{P}_m$. It is clear that any function $S \in \mathcal{P}_m$ can be added to f without changing the differential equation (7.8). However, if we require that f is orthogonal to the null space of $(\Delta^* + \lambda_0) \ \ldots \ (\Delta^* + \lambda_m)$, then the differential equation is uniquely solvable:

Let g be a continuous function on Ω orthogonal to $\mathcal{P}_m$, i.e.

$$\int_{\Omega} g(\eta)\ S_{n,j}(\eta)\ d\omega = 0 \tag{7.10}$$

for $n = 0, \dots, m$; $j = 1, \dots, 2n+1$.

Then the function f given by

$$f(\xi) = \left(-\frac{1}{4\pi}\right)^{m+1} \int_{\Omega} \mathcal{G}(\lambda_0, \dots, \lambda_m; \xi, \eta)\ g(\eta)\ d\omega(\eta) \tag{7.11}$$

represents the only (2m + 2) - times continuously differentiable solution of tne differential equation (7.8) which is orthogonal to $\mathcal{P}_m$.

For integers $m \geq 1$ the bilinear expansion

$$\mathcal{G}(\lambda_0, \dots, \lambda_m; \xi, \eta) = (4\pi)^m \sum_{k=m+1}^{\infty} \frac{2k+1}{(\lambda_k - \lambda_0)\dots(\lambda_k - \lambda_m)} P_k(\xi \cdot \eta)$$

is absolutely and uniformly convergent on the unit sphere Ω. Interchanging sum and integral we therefore obtain

$$f(\xi) = \frac{(-1)^{m+1}}{4\pi} \sum_{k=m+1}^{\infty} \frac{2k+1}{(\lambda_k - \lambda_0)\dots(\lambda_k - \lambda_m)} \int_{\Omega} g(\eta)\ P_k(\xi \cdot \eta)\ d\omega(\eta)$$

as unique solution of the differential equation (7.8). □

8. THE HILBERT SPACE $\mathcal{H}^{(2m+2)}(\Omega)$

In the class $C^{(2m+2)}(\Omega)$ of all $(2m+2)$ - times continuously differentiable functions on the sphere Ω we introduce the inner product

$$(f, g)_m \tag{8.1}$$

$$= \sum_{n=0}^{m} \sum_{j=1}^{2n+1} \left(\sqrt{\frac{2n+1}{4\pi}} \int_{\Omega} f(\eta) S_{n,j}(\eta)\, d\omega \right) \left(\sqrt{\frac{2n+1}{4\pi}} \int_{\Omega} g(\eta) S_{n,j}(\eta)\, d\omega \right)$$

$$+ \left(\frac{1}{4\pi} \right)^{2m+2} \int_{\Omega} \left[(\Delta_\eta^* + \lambda_0) \dots (\Delta_\eta^* + \lambda_m) f(\eta) \right] \left[(\Delta_\eta^* + \lambda_0) \dots (\Delta_\eta^* + \lambda_m) g(\eta) \right] d\omega .$$

The space $C^{(2m+2)}(\Omega)$ equipped with the inner product $(\cdot, \cdot)_m$ is a non-complete inner product space. For the sake of simplicity we use the abbreviation

$$K_m(\xi, \eta) = \sum_{n=0}^{m} P_n(\xi \cdot \eta) + \mathcal{G}^{(2)}(\lambda_0, \dots, \lambda_m; \xi, \eta) . \tag{8.2}$$

The kernel $K_m(\xi, \eta)$ admits the series expansion

$$K_m(\xi, \eta) = \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} Y_{n,j}(\xi)\, Y_{n,j}(\eta), \tag{8.3}$$

where the functions $Y_{n,j}$ are defined as follows

$$Y_{n,j} = \begin{cases} \sqrt{\frac{4\pi}{2n+1}}\, S_{n,j} & n = 0, \dots, m \\ & j = 1, \dots, 2n+1 \\ \\ \frac{(-4\pi)^{m+1}}{(\lambda_n - \lambda_0) \dots (\lambda_n - \lambda_m)}\, S_{n,j} & n = m+1, m+2, \dots \\ & j = 1, \dots, 2n+1 . \end{cases} \tag{8.4}$$

$K_m(\xi,\eta)$ has the representation of an isotropic covariance function on the unit sphere Ω (with respect to Legendre polynomials). It is clear that $K_m(\xi,\eta)$ is symmetric in the arguments ξ and η, $K_m(\xi,\eta) = K_m(\eta,\xi)$. By the Hecke formula we have

$$\int_\Omega P_n(\xi\cdot\eta)\, S_{k,j}(\eta)\, d\omega(\eta) = \frac{4\pi}{2n+1}\, S_{k,j}(\xi)\, \delta_{nk} \qquad (8.6)$$

$$= \begin{cases} 0 & n \neq k \\ \frac{4\pi}{2n+1}\, S_{n,j}(\xi) & n = k\,. \end{cases}$$

Consequently, an easy calculation shows that

$$(f, K_m(\xi,\cdot))_m \qquad (8.7)$$

$$= \sum_{n=0}^{m} \sum_{j=1}^{2n+1} S_{n,j}(\xi) \int_\Omega f(\eta)\, S_{n,j}(\eta)\, d\omega(\eta)$$

$$+ \left(\frac{1}{4\pi}\right)^{2m+2} \int_\Omega \left[(\Delta_\eta^* + \lambda_0)\cdots(\Delta_\eta^* + \lambda_m)\xi^{(2)}(\lambda_0,\ldots,\lambda_m;\xi,\eta)\right](\Delta_\eta^* + \lambda_0)\cdots(\Delta_\eta^* + \lambda_m) f(\eta)\, d\omega(\eta).$$

But this means that

$$f(\xi) = (f, K_m(\xi,\cdot))_m \qquad (8.8)$$

for every function $f \in C^{(2m+2)}(\Omega)$ and every point $\xi \in \Omega$.

$K_m(\xi,\eta)$ is a "reproducing kernel" in $C^{(2m+2)}(\Omega)$. However, the kernel $K_m(\xi,\eta)$ itself does not belong to $C^{(2m+2)}(\Omega)$ as a function of η for fixed ξ (or as a function of ξ for fixed η). And that is the reason why we have to complete our space $C^{(2m+2)}(\Omega)$ with

respect to the topology induced by $(\cdot\,,\cdot)_m$.

Let $\mathcal{H}^{(2m+2)}(\Omega)$ denote the space of all functions f satisfying

$$(f,f)_m = \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} |(f, Y_{n,j})_m|^2 < \infty . \qquad (8.9)$$

Then

$$K_m(\xi,\eta) = \sum_{n=0}^{m} P_n(\xi\cdot\eta) + \mathcal{G}^{(2)}(\lambda_0,\dots,\lambda_m;\xi,\eta) \qquad (8.10)$$

is the reproducing kernel for the Hilbert space $\mathcal{H}^{(2m+2)}(\Omega)$, i.e.:

(i) For each fixed $\xi \in \Omega$, $K_m(\xi,\eta)$ considered as function of η, is in $\mathcal{H}^{(2m+2)}(\Omega)$.

(ii) For every function $f \in \mathcal{H}^{(2m+2)}(\Omega)$ and every point $\xi \in \Omega$ the reproducing property

$$f(\xi) = (f, K_m(\xi,\cdot))_m \qquad (8.11)$$

is valid.

The system $\{Y_{n,j}\}_{\substack{n=0,1,\dots \\ j=1,\dots,2n+1}}$ is a Hilbert basis in $\mathcal{H}^{(2m+2)}(\Omega)$.

NOTE: For the case m = 0 see Freeden (1978/1980) and Krarup (1979).

In the sequel we shall analyze in detail that a proper abstract setting for the problems of interpolation by spherical splines will be

provided by the Hilbert space $\mathcal{H}^{(2m+2)}(\Omega)$ under consideration. □

For that purpose, we have to prove first that $\mathcal{H}^{(2m+2)}(\Omega)$ is a Hilbert function space of continuous functions on the unit sphere.

Let f be an element of $\mathcal{H}^{(2m+2)}(\Omega)$. Then, by Schwarz's inequality, we get with a suitable positive constant $C_m := \sqrt{K_m(\xi,\xi)}$ dependent on m

$$|f(\xi)| \leq C_m \sqrt{(f,f)_m} \tag{8.12}$$

uniformly with respect to all points $\xi \in \Omega$. Furthermore, there exists a sequence $\{f_k\}$ of elements f_k in $C^{(2m+2)}(\Omega)$ with $f_k \longrightarrow f$ (in the sense of the norm $(\cdot,\cdot)_m$). Therefore, in connection with (8.12), we obtain

$$|f_k(\xi) - f_n(\xi)| \leq C_m \sqrt{(f_k - f_n, f_k - f_n)_m} \tag{8.13}$$

uniformly with respect to all $\xi \in \Omega$. But this implies that $f_k(\xi) \longrightarrow f(\xi)$ uniformly with respect to $\xi \in \Omega$. Consequently, f is a continuous function on the sphere Ω. □

The Hilbert space $\mathcal{H}^{(2m+2)}(\Omega)$ is naturally equipped with the (Sobolev-like) semi-inner product $\langle \cdot, \cdot \rangle_m$ defined by

$$\langle f, g \rangle_m = \left(\frac{1}{4\pi}\right)^{2m+2} \int_{\Omega} \left[(\Delta_\eta^* + \lambda_0)\cdots(\Delta_\eta^* + \lambda_m) f(\eta)\right]\left[(\Delta_\eta^* + \lambda_0)\cdots(\Delta_\eta^* + \lambda_m) g(\eta)\right] d\omega \tag{8.14}$$

corresponding to the semi-norm

$$\|f\|_m = \left(\frac{1}{4\pi}\right)^{m+1} \sqrt{\int_{\Omega} |(\Delta_\eta^* + \lambda_0)\cdots(\Delta_\eta^* + \lambda_m) f(\eta)|^2 \, d\omega}, \tag{3.15}$$

where $(\Delta^*)_m = (\Delta^* + \lambda_0) \ldots (\Delta^* + \lambda_m)$ is to be interpreted in the distributional sense.

□

The null space of the semi-norm $\| \cdot \|_m = \sqrt{\langle \cdot, \cdot \rangle_m}$ is known to be simply the linear space $\mathcal{P}_m$ of dimension $M = (m+1)^2$.

It should be noted that the norm may be physically interpreted (at least for $m = 0$) as the bending energy of a (thin) membrane spanned wholly over the unit sphere Ω, f denotes the deflection normal to the rest position of course to be spherical. This model is suggested by the classical interpretation of the integral

$$\int_a^b |f''(x)|^2 \, dx$$

as the potential energy of a statically deflected thin beam which indeed is proportional to the integral taken over the square of the (linearized) curvature of the elastica of the beam (cf. Freeden (1981b)).

We summarize our results as follows: The (semi-normed) space $\mathcal{H}^{(2m+2)}(\Omega)$ defined by (8.9) and (8.15) is a semi-Hilbert space of continuous functions on the sphere Ω.

□

9. THE HILBERT SPACE $\mathcal{H}_0^{(2m+2)}(\Omega)$

Let $\eta_1, \ldots, \eta_M$ be a fundamental system of order m on Ω. Then there exists a unique $S \in \mathcal{P}_m$ satisfying

$$S(\eta_k) = \gamma_k \quad , \quad k = 1, \ldots, M, \tag{9.1}$$

for any prescribed (real) scalars $\gamma_1, \ldots, \gamma_M$. For, as function of $\mathcal{P}_m$, S is representable in the form

$$S(\eta) = \sum_{n=0}^{m} \sum_{j=1}^{2n+1} c_{n,j} \, S_{n,j}(\eta). \tag{9.2}$$

Substitution with $\eta = \eta_k$, $k = 1, \ldots, M$, gives M linear equations in these M coefficients $c_{n,j}$

$$\begin{aligned} \sum_{n=0}^{m} \sum_{j=1}^{2n+1} c_{n,j} \, S_{n,j}(\eta_1) &= \gamma_1 \\ \cdot \qquad\qquad\qquad &\quad \cdot \\ \cdot \qquad\qquad\qquad &\quad \cdot \\ \cdot \qquad\qquad\qquad &\quad \cdot \\ \sum_{n=0}^{m} \sum_{j=1}^{2n+1} c_{n,j} \, S_{n,j}(\eta_M) &= \gamma_M \, . \end{aligned} \tag{9.3}$$

The linear equations are uniquely solvable, since their matrix (cf. (4.6)) is assumed to be of full rank for a fundamental system of order m.

Hence, given a fundamental system $\eta_1, \ldots, \eta_M$ of order m on Ω, then there are determined uniquely M linearly independent elements of $\mathcal{P}_m$, $\mathcal{B}_1, \ldots, \mathcal{B}_M$ such that

$$\mathcal{B}_i(\eta_k) = \sum_{n=0}^{m} \sum_{j=1}^{2n+1} c_{n,j}^{i} \, S_{n,j}(\eta_k) = \delta_{ik} \tag{9.4}$$

i.e.:

$$\mathcal{B}_1(\eta_1) = 1 \;,\; \mathcal{B}_1(\eta_2) = 0 \;,\; \ldots \;,\; \mathcal{B}_1(\eta_M) = 0$$

$$\mathcal{B}_2(\eta_1) = 0 \;,\; \mathcal{B}_2(\eta_2) = 1 \;,\; \ldots \;,\; \mathcal{B}_2(\eta_M) = 0$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$\mathcal{B}_M(\eta_1) = 0 \;,\; \mathcal{B}_M(\eta_2) = 0 \;,\; \ldots \;,\; \mathcal{B}_M(\eta_M) = 1.$$

For any $S \in \mathcal{P}_m$ we have

$$S(\eta) = \sum_{k=1}^{M} S(\eta_k) \, \mathcal{B}_k(\eta) \tag{9.5}$$

as the (uniquely determined) $\mathcal{P}_m$ - interpolant of S on the point system $\eta_1, \ldots, \eta_M$.

For any choice of (real) values $y_1, \ldots, y_M$, the element

$$S(\eta) = \sum_{k=1}^{M} y_k \, \mathcal{B}_k(\eta) \tag{9.6}$$

is the unique solution of the interpolation problem $S(\eta_k) = y_k$, $k = 1, \ldots, M$, in $\mathcal{P}_m$.

The polynomials $\mathcal{B}_1, \ldots, \mathcal{B}_M$ of the class $\mathcal{P}_m$ are called the fundamental polynomials for pointwise interpolation. In order to evaluate the functions $\mathcal{B}_i$ given by

$$\mathcal{B}_i(\eta) = \sum_{n=0}^{m} \sum_{j=1}^{2n+1} c_{n,j}^{\,i}\, S_{n,j}(\eta)$$

corresponding to a prescribed fundamental system $\eta_1, \ldots, \eta_M$ of order m on Ω we have to solve the linear system

$$\begin{pmatrix} S_{0,1}(\eta_1) & \ldots & S_{m,1}(\eta_1) \ldots S_{m,2m+1}(\eta_1) \\ \vdots & & \vdots \\ S_{0,1}(\eta_M) & \ldots & S_{m,1}(\eta_M) \ldots S_{m,2m+1}(\eta_M) \end{pmatrix} \begin{pmatrix} c_{0,1}^{1} & \ldots & c_{0,1}^{M} \\ \vdots & & \vdots \\ c_{m,2m+1}^{1} & \ldots & c_{m,2m+1}^{M} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

i.e. the matrix

$$\begin{pmatrix} c_{0,1}^{1} & \cdots & c_{0,1}^{M} \\ \vdots & & \vdots \\ c_{m,1}^{1} & \cdots & c_{m,1}^{M} \\ \vdots & & \vdots \\ c_{m,2m+1}^{1} & \cdots & c_{m,2m+1}^{M} \end{pmatrix}$$

is equal to

$$(\alpha^M)^{-t} = ((\alpha^M)^t)^{-1} \tag{9.7}$$

($(\quad)^t$: transposed matrix).

For every $f \in \mathcal{H}^{(2m+2)}(\Omega)$,

$$\ell_f(\eta) = \sum_{k=1}^{M} f(\eta_k)\, B_k(\eta) \tag{9.8}$$

is called the (generalized) Lagrange form. The mapping

$$\ell_f : \mathcal{H}^{(2m+2)}(\Omega) \longrightarrow \mathcal{P}_m, \; f \longmapsto \ell_f$$

is a linear projector of $\mathcal{H}^{(2m+2)}(\Omega)$ with range $\mathcal{P}_m$ and kernel

$$\mathcal{H}_0^{(2m+2)}(\Omega) = \{ f \in \mathcal{H}^{(2m+2)} \mid f(\eta_k) = 0 \,,\ k = 1, \dots, M \}. \tag{9.9}$$

Each function $f \in \mathcal{H}^{(2m+2)}(\Omega)$ can be represented uniquely in the form

$$f(\eta) = \ell_f(\eta) + f_0(\eta) \,, \tag{9.10}$$

where f_0 is of class $\mathcal{H}_0^{(2m+2)}(\Omega)$ and f_0 and ℓ_f are orthogonal in the sense of the inner product $\langle \cdot \,, \cdot \rangle_m$:

$$\langle f_0, \ell_f \rangle_m \tag{9.11}$$

$$= \left(\frac{1}{4\pi}\right)^{2m+2} \int_{\Omega} \left[(\Delta_\eta^* + \lambda_0) \dots (\Delta_\eta^* + \lambda_m) f_0(\eta)\right]\left[(\Delta_\eta^* + \lambda_0) \dots (\Delta_\eta^* + \lambda_m) \ell_f(\eta)\right] d\omega$$

$$= 0 \,. \qquad \square$$

Equipped with the semi-norm

$$\| \cdot \|_m = \sqrt{\langle \cdot \,, \cdot \rangle_m}$$

the linear space $\mathcal{H}_0^{(2m+2)}(\Omega)$ is a Hilbert space of continuous functions on the unit sphere Ω. In view of the reproducing property of the kernel $K_m(\xi, \eta)$ in $\mathcal{H}^{(2m+2)}(\Omega)$ it is easy to see that

$$< f_0 \,,\, \mathcal{G}^{(2)}(\lambda_0, \dots, \lambda_m; \xi, \cdot) >_m \qquad (9.12)$$

$$- < f_0 \,,\, \sum_{n=1}^{M} \mathcal{G}^{(2)}(\lambda_0, \dots, \lambda_m; \xi, \eta_n)\, \mathcal{B}_n >_m$$

$$- < f_0 \,,\, \sum_{n=1}^{M} \mathcal{B}_n(\xi)\, \mathcal{G}^{(2)}(\lambda_0, \dots, \lambda_m; \eta_n, \cdot) >_m$$

$$+ < f_0 \,,\, \sum_{n=1}^{M} \sum_{k=1}^{M} \mathcal{B}_k(\xi)\, \mathcal{G}^{(2)}(\lambda_0, \dots, \lambda_m; \eta_n, \eta_k)\, \mathcal{B}_n >_m$$

$$= f_0(\xi)$$

for every function $f_0 \in \mathcal{H}_0^{(2n+2)}(\Omega)$ and every point $\xi \in \Omega$.

Moreover, the kernel

$$K_m^0(\xi, \eta) = \mathcal{G}^{(2)}(\lambda_0, \dots, \lambda_m; \xi, \eta) \qquad (9.13)$$

$$- \sum_{k=1}^{M} \mathcal{G}^{(2)}(\lambda_0, \dots, \lambda_m; \xi, \eta_k)\, \mathcal{B}_k(\eta)$$

$$- \sum_{k=1}^{M} \mathcal{B}_k(\xi)\, \mathcal{G}^{(2)}(\lambda_0, \dots, \lambda_m; \eta_k, \eta)$$

$$+ \sum_{k=1}^{M} \sum_{n=1}^{M} \mathcal{B}_k(\xi)\, \mathcal{G}^{(2)}(\lambda_0, \dots, \lambda_m; \eta_k, \eta_n)\, \mathcal{B}_n(\eta)$$

belongs to the space $\mathcal{H}_0^{(2m+2)}(\Omega)$.

Therefore, our considerations can be summarized as follows:

The linear space $\mathcal{H}_0^{(2m+2)}(\Omega)$ defined in (9.9) is equipped with the norm $\|\cdot\|_m = \sqrt{\langle \cdot, \cdot \rangle_m}$ a Hilbert space. $\mathcal{H}_0^{(2m+2)}(\Omega)$ possesses the reproducing kernel (9.13).

□

10. OPTIMAL INTERPOLATION PROBLEM

The basic problem of spherical interpolation shall be repeated now explicitly as follows:

- given an admissible system $\eta_1, \dots, \eta_N$ of points of the unit sphere Ω and a set $y_1, \dots, y_N$ of real scalars,
- construct a function $s : \Omega \longrightarrow \mathbb{R}$ belonging to the linear variety

$$V_N = \{ f \in \mathcal{H}^{(2m+2)}(\Omega) \mid f(\eta_k) = y_k, \; k = 1, \dots, N \}$$

of all interpolants in $\mathcal{H}^{(2m+2)}(\Omega)$ to the data.

To assure uniqueness of this problem, additional information is clearly necessary. In problems of determining geodetic or geophysical quantities some restrictions on smoothness and polynomial precision are required. This can be achieved by restricting the set of interpolants using an (energy) semi-norm. An interpolant minimizing such a semi-norm can be regarded as the "smoothest" solution. In this connection the semi-norm $\| \cdot \|_m$ represents a natural setting as explained in our introduction.

From the definition of $\| \cdot \|_m$ it is clear that

$$\| \mathcal{B}_n \|_m = \left(\frac{1}{4\pi}\right)^{m+1} \sqrt{\int_\Omega \left| (\Delta_\eta^* + \lambda_0) \dots (\Delta_\eta^* + \lambda_m) \, \mathcal{B}_n(\eta) \right|^2 d\omega} \qquad (10.2)$$

$$= 0$$

for $n = 1, \dots, M$.

Hence, the minimization problem $(N > M)$

$$\left\{ \begin{array}{l} \| s \|_m = \inf\limits_{f \in V_N} \| f \|_m \\ V_N = \{ f \in \mathcal{H}^{(2m+2)}(\Omega) \mid f(\eta_i) = y_i, \; i = 1, \dots, N \} \end{array} \right\} \qquad (10.3)$$

is equivalent to the problem

$$\left\{ \begin{array}{l} \| s_0 \|_m = \inf_{f_0 \in V_N^0} \| f_0 \|_m \\ V_N^0 = \{ f_0 \in \mathcal{H}_0^{(2m+2)}(\Omega) \mid f_0(\eta_j) = y_j, j = M+1, \ldots, N \} \end{array} \right\}. \quad (10.4)$$

Remark: The case N = M reduces to strict interpolation by spherical harmonics.

From Chapter 9 we know that $K_m^0(\xi, \eta)$ is the reproducing kernel of $\mathcal{H}_0^{(2m+2)}(\Omega)$. Thus, for $\xi, \eta \in \Omega$,

$$< K_m^0(\xi, \cdot), K_m^0(\cdot, \eta) >_m \quad (10.5)$$

$$= \left(\frac{1}{4\pi}\right)^{2m+2} \int_\Omega [(\Delta_\zeta^* + \lambda_0) \ldots (\Delta_\zeta^* + \lambda_m) K_m^0(\xi, \zeta)][(\Delta_\zeta^* + \lambda_0) \ldots (\Delta_\zeta^* + \lambda_m) K_m^0(\zeta, \eta)] \, d\omega(\zeta)$$

$$= K_m^0(\xi, \eta).$$

But this means that the matrix

$$\sigma_M^N = \begin{pmatrix} K_m^0(\eta_{M+1}, \eta_{M+1}) & \cdots & K_m^0(\eta_{M+1}, \eta_N) \\ \vdots & & \vdots \\ K_m^0(\eta_N, \eta_{M+1}) & \cdots & K_m^0(\eta_N, \eta_N) \end{pmatrix} \quad (10.6)$$

is symmetric and positive definite, viz. as Gram matrix of the (linearly independent) functions $K_m^0(\eta_{M+1}, \eta), \ldots, K_m^0(\eta_N, \eta)$.

These results now can be used to prove the main theorem:

Suppose that $(\eta_1, y_1), \ldots, (\eta_N, y_N)$ are prescribed data points

corresponding to an admissible system $\eta_1, \ldots, \eta_N$ of order m on the unit sphere Ω. Then, the function $s \in \mathcal{H}^{(2m+2)}(\Omega)$ given by

$$s(\eta) = \ell_s(\eta) + s_0(\eta) \tag{10.7}$$

with

$$\ell_s(\eta) = \sum_{n=1}^{M} \gamma_N \, B_n(\eta) \tag{10.8}$$

and

$$s_0(\eta) = \sum_{k=M+1}^{N} a_k \, K_m^{\circ}(\eta_k, \eta) \tag{10.9}$$

is the only solution of the interpolation problem

$$\|s\|_m = \inf_{f \in V_N} \|f\|_m ,$$

i.e.:

$$\left(\frac{1}{4\pi}\right)^{2m+2} \int_{\Omega} |(\Delta_\eta^* + \lambda_0) \ldots (\Delta_\eta^* + \lambda_m) \, s(\eta)|^2 \, d\omega$$

$$- \inf_{f \in V_N} \left(\frac{1}{4\pi}\right)^{2m+2} \int_{\Omega} |(\Delta_\eta^* + \lambda_0) \ldots (\Delta_\eta^* + \lambda_m) \, f(\eta)|^2 \, d\omega ,$$

where the coefficients $a_{M+1}, \ldots, a_N$ satisfy

the linear equations

$$\sum_{k=M+1}^{N} a_k \, K_m^\circ(\eta_k, \eta_j) = y_j - \sum_{n=1}^{M} y_n \, \mathcal{B}_n(\eta_j) \qquad (10.10)$$

$$(j = M+1, \ldots, N).$$

Proof. (Uniqueness) Suppose that s_0 is given in the form (10.9). Then s_0 is uniquely determined, since the linear equations (10.10) admit one and only one solution $(a_{M+1}, \cdots, a_N)$ because of the positive definiteness of (10.6).

(Minimum property) In order to prove the minimum property of s_0 in V_N° with respect to $\| \cdot \|_m$ we consider the difference d_0 of s_0 and any function $f_0 \in V_N^\circ$:

$$d_0 = s_0 - f_0 . \qquad (10.11)$$

Of course, we have $d_0(\eta_k) = 0$ for $k = M+1, \ldots, N$. Hence, by virtue of the reproducing property of $K_m^\circ(\xi, \eta)$ in $\mathcal{H}_0^{(2m+2)}(\Omega)$ it follows that

$$\left(\frac{1}{4\pi}\right)^{2m+2} \int_\Omega \left[(\Delta_\eta^* + \lambda_0) \cdots (\Delta_\eta^* + \lambda_m) \, d_0(\eta)\right]\left[(\Delta_\eta^* + \lambda_0) \cdots (\Delta_\eta^* + \lambda_m) \, s_0(\eta)\right] d\omega \qquad (10.12)$$

$$= \left(\frac{1}{4\pi}\right)^{2m+2} \sum_{k=M+1}^{N} a_k \int_\Omega \left[(\Delta_\eta^* + \lambda_0) \cdots (\Delta_\eta^* + \lambda_m) \, d_0(\eta)\right]\left[(\Delta_\eta^* + \lambda_0) \cdots (\Delta_\eta^* + \lambda_m) \, K_m^\circ(\eta_k, \eta)\right] d\omega$$

$$= \sum_{k=M+1}^{N} a_k \, d_0(\eta_k) = 0 .$$

Therefore it is obvious that

$$\left(\frac{1}{4\pi}\right)^{2m+2}\int_{\Omega} |(\Delta_\eta^{*}+\lambda_0)\dots(\Delta_\eta^{*}+\lambda_m) f_0(\eta)|^2 \, d\omega(\eta) \qquad (10.13)$$

$$= \left(\frac{1}{4\pi}\right)^{2m+2}\int_{\Omega} |(\Delta_\eta^{*}+\lambda_0)\dots(\Delta_\eta^{*}+\lambda_m)(s_0(\eta)-d_0(\eta))|^2 \, d\omega(\eta)$$

$$= \left(\frac{1}{4\pi}\right)^{2m+2}\int_{\Omega} |(\Delta_\eta^{*}+\lambda_0)\dots(\Delta_\eta^{*}+\lambda_m) s_0(\eta)|^2 \, d\omega(\eta)$$

$$+ \left(\frac{1}{4\pi}\right)^{2m+2}\int_{\Omega} |(\Delta_\eta^{*}+\lambda_0)\dots(\Delta_\eta^{*}+\lambda_m) d_0(\eta)|^2 \, d\omega(\eta) .$$

This shows that

$$\left(\frac{1}{4\pi}\right)^{2m+2}\int_{\Omega} |(\Delta_\eta^{*}+\lambda_0)\dots(\Delta_\eta^{*}+\lambda_m) s_0(\eta)|^2 \, d\omega(\eta)$$

$$\le \left(\frac{1}{4\pi}\right)^{2m+2}\int_{\Omega} |(\Delta_\eta^{*}+\lambda_0)\dots(\Delta_\eta^{*}+\lambda_m) f_0(\eta)|^2 \, d\omega(\eta)$$

for all $f_0 \in V_N^0$, where equality holds if and only if $d_0 = 0$, i.e. $f_0 = s_0$. □

For later use we write the matrix

$$\sigma_M^N = \begin{pmatrix} K_m^0(\eta_{M+1},\eta_{M+1}) & \dots & K_m^0(\eta_{M+1},\eta_N) \\ \vdots & & \vdots \\ K_m^0(\eta_N,\eta_{M+1}) & \dots & K_m^0(\eta_N,\eta_N) \end{pmatrix}$$

in decomposed form. Observing the explicit representation of (see (9.13)) we obtain

$$G_M^N = \begin{pmatrix} g^{(2)}(\lambda_0,\dots,\lambda_m;\gamma_{M+1},\gamma_{M+1}) & \cdots & g^{(2)}(\lambda_0,\dots,\lambda_m;\gamma_{M+1},\gamma_N) \\ \vdots & & \vdots \\ g^{(2)}(\lambda_0,\dots,\lambda_m;\gamma_N,\gamma_{M+1}) & \cdots & g^{(2)}(\lambda_0,\dots,\lambda_m;\gamma_N,\gamma_N) \end{pmatrix} \tag{10.14}$$

$$- \begin{pmatrix} g^{(2)}(\lambda_0,\dots,\lambda_m;\gamma_{M+1},\gamma_1) & \cdots & g^{(2)}(\lambda_0,\dots,\lambda_m;\gamma_{M+1},\gamma_M) \\ \vdots & & \vdots \\ g^{(2)}(\lambda_0,\dots,\lambda_m;\gamma_N,\gamma_1) & \cdots & g^{(2)}(\lambda_0,\dots,\lambda_m;\gamma_N,\gamma_M) \end{pmatrix} \begin{vmatrix} B_1(\gamma_{M+1}) & \cdots & B_1(\gamma_N) \\ \vdots & & \vdots \\ B_M(\gamma_{M+1}) & \cdots & B_M(\gamma_N) \end{vmatrix}$$

$$- \begin{vmatrix} B_1(\gamma_{M+1}) & \cdots & B_M(\gamma_{M+1}) \\ \vdots & & \vdots \\ B_1(\gamma_N) & \cdots & B_M(\gamma_N) \end{vmatrix} \begin{pmatrix} g^{(2)}(\lambda_0,\dots,\lambda_m;\gamma_{M+1},\gamma_1) & \cdots & g^{(2)}(\lambda_0,\dots,\lambda_m;\gamma_N,\gamma_1) \\ \vdots & & \vdots \\ g^{(2)}(\lambda_0,\dots,\lambda_m,\gamma_{M+1},\gamma_M) & \cdots & g^{(2)}(\lambda_0,\dots,\lambda_m;\gamma_N,\gamma_M) \end{pmatrix}$$

$$+\begin{vmatrix} \mathcal{B}_1(\eta_{M+1}) \dots \mathcal{B}_M(\eta_{M+1}) \\ \cdot \quad\quad \cdot \\ \cdot \quad\quad \cdot \\ \cdot \quad\quad \cdot \\ \mathcal{B}_1(\eta_N) \dots \mathcal{B}_M(\eta_N) \end{vmatrix} \cdot \begin{vmatrix} g^{(2)}(\lambda_0,\dots,\lambda_m;\eta_1,\eta_1) \dots g^{(2)}(\lambda_0,\dots,\lambda_m;\eta_1,\eta_M) \\ \cdot \quad\quad\quad\quad \cdot \\ \cdot \quad\quad\quad\quad \cdot \\ \cdot \quad\quad\quad\quad \cdot \\ g^{(2)}(\lambda_0,\dots,\lambda_m;\eta_M,\eta_1) \dots g^{(2)}(\lambda_0,\dots,\lambda_m;\eta_M,\eta_M) \end{vmatrix} \begin{vmatrix} \mathcal{B}_1(\eta_{M+1}) \dots \mathcal{B}_1(\eta_N) \\ \cdot \quad\quad \cdot \\ \cdot \quad\quad \cdot \\ \cdot \quad\quad \cdot \\ \mathcal{B}_M(\eta_{M+1}) \dots \mathcal{B}_M(\eta_N) \end{vmatrix} .$$

According to the definition of the fundamental functions $\mathcal{B}_1, \dots, \mathcal{B}_M$ of the class $\mathcal{P}_m$ we obtain

$$\begin{vmatrix} \mathcal{B}_1(\eta_{M+1}) \dots \mathcal{B}_M(\eta_{M+1}) \\ \cdot \quad\quad \cdot \\ \cdot \quad\quad \cdot \\ \cdot \quad\quad \cdot \\ \mathcal{B}_1(\eta_N) \quad \dots \quad \mathcal{B}_M(\eta_N) \end{vmatrix} \tag{10.15}$$

$$= \begin{pmatrix} \sum_{n=0}^{m} \sum_{j=1}^{2n+1} c_{n,j}^{1} S_{n,j}(\eta_{M+1}) & \dots & \sum_{n=0}^{m} \sum_{j=1}^{2n+1} c_{n,j}^{M} S_{n,j}(\eta_{M+1}) \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \sum_{n=0}^{m} \sum_{j=1}^{2n+1} c_{n,j}^{1} S_{n,j}(\eta_N) & \dots & \sum_{n=0}^{m} \sum_{j=1}^{2n+1} c_{n,j}^{M} S_{n,j}(\eta_N) \end{pmatrix}$$

$$= \begin{pmatrix} S_{0,1}(\eta_{M+1}) & \cdots & S_{m,2m+1}(\eta_{M+1}) \\ \vdots & & \vdots \\ S_{0,1}(\eta_N) & \cdots & S_{m,2m+1}(\eta_N) \end{pmatrix} \begin{pmatrix} c_{0,1}^{1} & \cdots & c_{0,1}^{M} \\ \vdots & & \\ c_{m,1}^{1} & \cdots & c_{m,1}^{M} \\ \vdots & & \vdots \\ c_{m,2m+1}^{1} & \cdots & c_{m,2m+1}^{M} \end{pmatrix}.$$

The matrix

$$\begin{pmatrix} c_{0,1}^{1} & \cdots & c_{0,1}^{M} \\ \vdots & & \vdots \\ c_{m,1}^{1} & \cdots & c_{m,1}^{M} \\ \vdots & & \vdots \\ c_{m,2m+1}^{1} & \cdots & c_{m,2m+1}^{M} \end{pmatrix}$$

is known from (9.7) to be equal to $(\alpha^M)^{-t}$.

Consequently,

$$\begin{pmatrix} \mathcal{B}_1(\eta_{M+1}) & \cdots & \mathcal{B}_M(\eta_{M+1}) \\ \vdots & & \vdots \\ \mathcal{B}_1(\eta_N) & \cdots & \mathcal{B}_M(\eta_N) \end{pmatrix} \tag{10.16}$$

$$= \begin{vmatrix} S_{0,1}(\gamma_{M+1}) & \cdots & S_{m,2m+1}(\gamma_{M+1}) \\ \vdots & & \vdots \\ S_{0,1}(\gamma_N) & \cdots & S_{m,2m+1}(\gamma_N) \end{vmatrix} \begin{vmatrix} S_{0,1}(\gamma_1) & \cdots & S_{m,2m+1}(\gamma_1) \\ \vdots & & \vdots \\ S_{0,1}(\gamma_M) & \cdots & S_{m,2m+1}(\gamma_M) \end{vmatrix}^{-1} .$$

□

11. SPLINE FUNCTIONS

The consideration given in Chapter 10 now leads us to the definition of a spherical spline with respect to an admissible system of order m on Ω :

Let $\eta_1, \ldots, \eta_N$ be an admissible system of order m on the unit sphere Ω . Then any function $s \in \mathcal{H}^{(2m+2)}(\Omega)$ of the form (10.7)

$$s(\eta) = \sum_{n=1}^{M} \gamma_n B_n(\eta) + \sum_{k=M+1}^{N} a_k K_m^{\circ}(\eta_k, \eta) \tag{11.1}$$

with arbitrarily given reals $\gamma_1, \ldots, \gamma_N$ and coefficients $a_{M+1}, \ldots, a_N$ is called a spherical spline function in $\mathcal{H}^{(2m+2)}(\Omega)$ relative to $\eta_1, \ldots, \eta_N$.

The class of all spline functions s in $\mathcal{H}^{(2m+2)}(\Omega)$ relative to $\eta_1, \ldots, \eta_N$ is denoted by $\mathcal{S}_m(\eta_1, \ldots, \eta_N)$.

The space $\mathcal{S}_m(\eta_1, \ldots, \eta_N)$ is a N - dimensional linear subspace of $\mathcal{H}^{(2m+2)}(\Omega)$ containing the class $\mathcal{P}_m$.

From Chapter 10 it is known that the following interpolation property is valid in $\mathcal{H}^{(2m+2)}(\Omega)$:

Given $f \in \mathcal{H}^{(2m+2)}(\Omega)$, then there exists a unique element $s \in \mathcal{S}_m(\eta_1, \ldots, \eta_N)$ satisfying

$$s(\eta_k) = f(\eta_k) = \gamma_k$$

for $k = 1, \ldots, N$. Denote this uniquely determined element of $\mathcal{S}_m(\eta_1, \ldots, \eta_N)$ by s_f. □

By virtue of the definition of a spherical spline function it is easy to show that the first integral relation

$$\left(\frac{1}{4\pi}\right)^{2m+2} \int_{\Omega} |(\Delta_\eta^* + \lambda_0) \cdots (\Delta_\eta^* + \lambda_m) f(\eta)|^2 d\omega \tag{11.2}$$

$$= \left(\frac{1}{4\pi}\right)^{2m+2} \int_{\Omega} |(\Delta_\eta^* + \lambda_0) \cdots (\Delta_\eta^* + \lambda_m)(f(\eta) - s_f(\eta))|^2 d\omega$$

$$+ \left(\frac{1}{4\pi}\right)^{2m+2} \int_{\Omega} |(\Delta_\eta^* + \lambda_0) \cdots (\Delta_\eta^* + \lambda_m) s_f(\eta)|^2 d\omega$$

holds for every $f \in \mathcal{H}^{(2m+2)}(\Omega)$. □

Moreover, it is easy to see that the second integral relation

$$\left(\frac{1}{4\pi}\right)^{2m+2} \int_{\Omega} |(\Delta_\eta^* + \lambda_0) \cdots (\Delta_\eta^* + \lambda_m)(f(\eta) - s(\eta))|^2 d\omega \tag{11.3}$$

$$= \left(\frac{1}{4\pi}\right)^{2m+2} \int_{\Omega} |(\Delta_\eta^* + \lambda_0) \cdots (\Delta_\eta^* + \lambda_m)(f(\eta) - s_f(\eta))|^2 d\omega$$

$$+ \left(\frac{1}{4\pi}\right)^{2m+2} \int_{\Omega} |(\Delta_\eta^* + \lambda_0) \cdots (\Delta_\eta^* + \lambda_m)(s(\eta) - s_f(\eta))|^2 d\omega$$

holds for every $f \in \mathcal{H}^{(2m+2)}(\Omega)$ and every $s \in \mathcal{S}_m(\eta_1, \ldots, \eta_N)$ (cf. Freeden 1981a).

□

In order to calculate an interpolating spline function

$$s_f(\eta) = \sum_{n=1}^{M} f(\eta_n) B_n(\eta) + \sum_{k=M+1}^{N} a_k K_m^\circ(\eta_k, \eta) \qquad (11.4)$$

we have to solve the linear equations

$$\sum_{k=M+1}^{N} a_k K_m^\circ(\eta_k, \eta_j) = f(\eta_j) - \sum_{n=1}^{M} f(\eta_n) B_n(\eta_j), \qquad (11.5)$$

$$(j = M+1, \ldots, N)$$

whose coefficient - matrix is symmetric and positive definite. Solving the system (11.5) is therefore a most simple problem: the matrix can be factorized by the well-known Cholesky procedure. This can be carried through without any need for pivoting or scaling (cf. Chapt. 13).

□

12. APPROXIMATION OF LINEAR FUNCTIONALS

Let us consider linear functionals $J : \mathcal{H}^{(2m+2)}(\Omega) \longrightarrow \mathbb{R}$ of the following structure

$$J f = \int_{\Omega} g(\eta) f(\eta) d\omega + \sum_{k=1}^{d} b_k f(\xi_k), \tag{12.1}$$

where the function g is assumed to be piecewise continuous on Ω, the weights $b_1, \ldots, b_d$ are real and the points $\xi_1, \ldots, \xi_d$ lie on Ω (d: positive integer).

Though this is not the most general class of functionals that might be considered on the Hilbert space $\mathcal{H}^{(2m+2)}(\Omega)$ it is adequate for most purposes and applications. Obviously included as special cases are the following:

(i) the integral over Ω (cf. Freeden (1978))

$$\int_{\Omega} f(\eta) d\omega \tag{12.1a}$$

or any subdomain Φ_f of Ω

$$\int_{\Phi_f} g(\eta) f(\eta) d\omega \tag{12.1b}$$

(ii) the orthogonal coefficients of a function f on Ω (cf. Freeden (1980/1981 a))

$$\int_{\Omega} f(\eta) S_{n,j}(\eta) d\omega$$

being required in connection with (Fourier) series expansions of a function into spherical harmonics

(iii) any functional value $f(\xi)$, $\xi \in \Omega$,

$$f(\xi) \tag{12.1d}$$

(iv) any finite linear combination of functional values of f at prescribed points $\xi_1, \ldots, \xi_d$ on Ω (cf. Freeden (1981a))

$$\sum_{k=1}^{d} b_k f(\xi_k) . \tag{12.1e}$$

Our purpose is to approximate a linear functional of the form (12.1) by a linear combination L of the form

$$Lf = \sum_{k=M+1}^{N} a_k \left[f(\eta_k) - \sum_{j=1}^{M} B_j(\eta_k) f(\eta_j) \right] , \tag{12.2}$$

where $\eta_1, \ldots, \eta_N$ is a given admissible system of order m on Ω.

The functional L is called <u>exact for the degree</u> m, if $LS = JS$ whenever $S \in \mathcal{P}_m$. The <u>remainder</u>, when L is used to approximate J, is a linear functional R defined by $R = J - L$. If the approximation of J by L is exact for the degree m, then $RS = 0$ whenever $S \in \mathcal{P}_m$.

Let f be a function of the space $\mathcal{H}^{(2m+2)}(\Omega)$ given in the form (9.10)

$$f(\eta) = \ell_f(\eta) + f_0(\eta), \tag{12.3}$$

where ℓ_f is of class $\mathcal{P}_m$ and f_0 is an element of $\mathcal{H}_0^{(2m+2)}(\Omega)$.

Then we first get

$$\begin{aligned} \mathcal{R} f &= \mathcal{R} \ell_f + \mathcal{R} f_0 \\ &= \mathcal{R} f_0 \end{aligned} \tag{12.4}$$

provided the approximation J by L is exact for the degree m. Moreover, the reproducing property of $\kappa_m^\circ(\xi, \eta)$ in $\mathcal{H}_0^{(2m+2)}(\Omega)$ implies that

$$\mathcal{R} f_0 = \left(\frac{1}{4\pi}\right)^{2m+2} \mathcal{R}_\xi \int_\Omega \left[(\Delta_\eta^* + \lambda_0)\dots(\Delta_\eta^* + \lambda_m) K_m^\circ(\xi,\eta)\right]\left[(\Delta_\eta^* + \lambda_0)\dots(\Delta_\eta^* + \lambda_m) f_0(\eta)\right] d\omega(\eta) \tag{12.5}$$

$$= \left(\frac{1}{4\pi}\right)^{2m+2} \int_\Omega \left[(\Delta_\eta^* + \lambda_0)\dots(\Delta_\eta^* + \lambda_m) \mathcal{R}_\xi K_m^\circ(\xi,\eta)\right]\left[(\Delta_\eta^* + \lambda_0)\dots(\Delta_\eta^* + \lambda_m) f_0(\eta)\right] d\omega(\eta).$$

(Observe: R is a bounded linear functional).

The notation $\mathcal{R}_\xi$ means the linear functional R is applied to the ξ variable of $\kappa_m^\circ(\xi, \eta)$. The function K given by

$$K(\eta) = \mathcal{R}_\xi K_m^\circ(\xi, \eta), \quad \eta \in \Omega, \tag{12.6}$$

is called the spherical Peano kernel of order m for the functional R.

Combining (12.4), (12.5) and (12.6) we therefore find in connection with (12.2)

$$Rf = \left(\frac{1}{4\pi}\right)^{2m+2} \int_{\Omega} \left[(\Delta_\zeta^* + \lambda_0)\dots(\Delta_\zeta^* + \lambda_m) K(\zeta)\right]\left[(\Delta_\zeta^* + \lambda_0)\dots(\Delta_\zeta^* + \lambda_m) f_0(\zeta)\right] d\omega \qquad (12.7)$$

$$= \left(\frac{1}{4\pi}\right)^{2m+2} \int_{\Omega} \left[(\Delta_\zeta^* + \lambda_0)\dots(\Delta_\zeta^* + \lambda_m) K(\zeta)\right]\left[(\Delta_\zeta^* + \lambda_0)\dots(\Delta_\zeta^* + \lambda_m) f(\zeta)\right] d\omega$$

with

$$K(\zeta) = R_\xi K_m^\circ(\xi,\zeta) \qquad (12.8)$$

$$= J_\xi K_m^\circ(\xi,\zeta) - L_\xi K_m^\circ(\xi,\zeta)$$

$$= J_\xi K_m^\circ(\xi,\zeta) - \sum_{k=m+1}^{N} a_k K_m^\circ(\zeta_k,\zeta).$$

Applying the Cauchy - Schwarz inequality to the right hand side of (12.7) we obtain

$$|Rf| \le \left(\frac{1}{4\pi}\right)^{m+1} \sqrt{\int_{\Omega} |(\Delta_\zeta^* + \lambda_0)\dots(\Delta_\zeta^* + \lambda_m) K(\zeta)|^2 d\omega} \qquad (12.9)$$

$$\cdot \sqrt{\int_{\Omega} |(\Delta_\zeta^* + \lambda_0)\dots(\Delta_\zeta^* + \lambda_m) f(\zeta)|^2 d\omega}\,.$$

We note that the quantity

$$\langle K, K \rangle_m = \left(\frac{1}{4\pi}\right)^{2m+2} \int_{\Omega} |(\Delta_\eta^* + \lambda_0) \dots (\Delta_\eta^* + \lambda_m) K(\eta)|^2 \, d\omega \qquad (12.10)$$

depends on the nodes $\eta_1, \dots, \eta_N$ and on the operator $\mathcal{R}$, but not on the function $f \in \mathcal{H}^{(2m+2)}(\Omega)$.

Collecting our results we obtain the following a priori estimate: let J be a linear functional of the form (12.1) and let L be any approximation of the form (12.2), exact for the degree m. Then, for each function $f \in \mathcal{H}^{(2m+2)}(\Omega)$,

$$|Jf - Lf| \le \left(\frac{1}{4\pi}\right)^{m+1} \sqrt{(J_\xi J_\eta - 2 J_\xi L_\eta + L_\xi L_\eta) K_m^\circ(\xi, \eta)} \qquad (12.11)$$

$$\cdot \sqrt{\int_{\Omega} |(\Delta_\eta^* + \lambda_0) \dots (\Delta_\eta^* + \lambda_m) f(\eta)|^2 d\omega}.$$

The estimate (12.11) enables us to calculate the best-approximation to J, i.e. the linear functional L of the form (12.2)

$$Lf = \sum_{k=M+1}^{N} a_k \left[f(\eta_k) - \sum_{j=1}^{M} B_j(\eta_k) f(\eta_j) \right] \qquad (12.12)$$

exact for the degree m, for which the quantity $\langle K, K \rangle_m$ assumes its minimum.

The minimum can be obtained by solving the uniquely determined quadratic optimization problem

$$J_\xi J_\eta K_m^\circ(\xi,\eta) - 2\sum_{k=M+1}^{N} a_k J_\xi K_m^\circ(\xi,\eta_k) + \sum_{n=M+1}^{N}\sum_{k=M+1}^{N} a_n a_k K_m^\circ(\eta_n,\eta_k)$$

$$\longrightarrow \quad \min. \qquad (12.13)$$

It is easy to see that the quadratic optimization problem becomes its minimum if

$$\sum_{k=M+1}^{N} a_k K_m^\circ(\eta_j,\eta_k) = J_\xi K_m^\circ(\xi,\eta_j) \qquad (12.14)$$

for $j = M+1, \ldots, N$. The linear equations (12.14), however, have a unique solution in the coefficients $a_{M+1}, \ldots, a_N$.

Summarizing our results we obtain the theorem:

Let J be a linear functional on $\mathcal{H}^{(2m+2)}(\Omega)$ of the form (12.1). Suppose that $(\tilde{a}_{M+1}, \ldots, \tilde{a}_N)$ is the solution of the linear equations (12.14). Then, for each $f \in \mathcal{H}^{(2m+2)}(\Omega)$, the linear functional L given by

$$Lf = \sum_{n=M+1}^{N} \tilde{a}_n \left[f(\eta_n) - \sum_{k=1}^{M} B_k(\eta_n) f(\eta_k) \right]$$

represents the best approximation to J (in the sense of the $\mathcal{L}^2$-estimate (12.9)).

The approximation L to J has as a posteriori estimate

$$|Jf - Lf|$$

$$\leq \left(\frac{1}{4\pi}\right)^{m+1} \sqrt{J_\xi J_\eta K_m^\circ(\xi,\eta) - \sum_{n=1+m}^{N} \tilde{a}_n J_\xi K_m^\circ(\xi,\eta_n)} \qquad (12.15)$$

$$\cdot \sqrt{\int_\Omega |(\Delta_\eta^* + \lambda_0) \ldots (\Delta_\eta^* + \lambda_m) f(\eta)|^2 \, d\omega} \, .$$

The best approximation ("F - approximation" in the sense of Krarup (1979)) opens a new (non-statistical) perspective for geodetic purposes of prediction: to supplement the gravity information, which can be made at only a relatively few points by determining further values of gravity at other points or to compute best approximations of other quantities of gravity. Furthermore, we have a priori information. □

From a theoretical, but also from a practical point of view it is of great interest that there is a close connection between interpolating spline and best approximation. This can be explained as follows (for a detailed proof see Freeden (1981a), Reuter (1982)):

Given a function $f \in \mathcal{H}^{(2m+2)}(\Omega)$. Denote by S_f the uniquely determined interpolating spline

$$|S_f|_m = \inf_{g \in V_N} |g|_m \qquad (12.16)$$

to the linear variety

$$V_N = \{ g \in \mathcal{H}^{(2m+2)}(\Omega) \mid g(\eta_j) = y_j = f(\eta_j), j = 1, \ldots, N \}.$$

Then the best approximation L to J is also uniquely determined by the property that $Lf = Js_f$ whenever $f \in \mathcal{H}^{(2m+2)}(\Omega)$.

In other words: there are two equivalent ways to compute the best approximation L to J

(i) solve the linear equations (12.14) (using Cholesky's factorization)

(ii) apply the functional J to the (optimal) interpolating spline s_f.

□

Remark: Approximation of integrals based on the idea of Weyl's law of uniform distribution can be found in Hlawka (1982).

13. NUMERICAL METHOD

Using generalized splines as e.g. proposed in Freeden (1981 a,b)(1982) and Meissl (1981), one deals with functions lacking a local support. Hence the normal equations are full and the size of the systems is limited in interpolation problems.

According to our considerations the data relative to various interpolation points now will be exploited in two steps based on a combination of well-known procedures in interpolation theory (cf. Davis (1963)), viz. the Lagrange interpolation and the Newton representation theorem. This is of great computational advantage as regards the number of data and the numerical effort. The standard algorithms to be used in our method (Cholesky's decomposition, forward substitution) are indeed very economical and numerically stable.

The great benefit of the method proposed here, however, is that the solution process can be formulated in a recursive way leading to the permanence property in spline interpolation problems. This will be discussed now.

As we have shown, the spherical spline $S \in \mathcal{H}^{(2m+2)}(\Omega)$ that interpolates the data points $(\eta_1, Y_1), \ldots, (\eta_N, Y_N)$ can be represented in the form

$$S(\eta) = \sum_{k=1}^{M} Y_k \mathcal{B}_k(\eta) + \sum_{n=M+1}^{N} a_n K_m^{\circ}(\eta_n, \eta). \qquad (13.1)$$

Assumed the points $\eta_1, \ldots, \eta_M$ form a fundamental system of order m on Ω the functions $\mathcal{B}_1, \ldots, \mathcal{B}_M$ satisfy the property

$$\mathcal{B}_n(\eta_k) = \delta_{nk} \qquad (13.2)$$

for $n, k = 1, \ldots, M$ (δ_{nk} : Kronecker symbol).
If $p \in \mathcal{P}_m$, then

$$p(\eta) = \sum_{n=1}^{M} p(\eta_n)\, \mathcal{B}_n(\eta) . \tag{13.3}$$

Given $y_1, \ldots, y_M$, the function defined by

$$p(\eta) = \sum_{n=1}^{M} y_n\, \mathcal{B}_n(\eta) \tag{13.4}$$

is the uniquely determined solution of the interpolation problem

$$p(\eta_n) = y_n \quad , \quad n = 1, \ldots, M \tag{13.5}$$

in the polynomial space $\mathcal{P}_m$.

The expression

$$p(\eta) = \sum_{n=1}^{M} p(\eta_n)\, \mathcal{B}_n(\eta) \tag{13.6}$$

is called (generalized) Lagrange formula.

As is well-known (cf. e.g. Davis (1963)), the Lagrange formula has one drawback, namely the lack of flexibility when passing from an interpolant to M data to an interpolant of more than M data. For instance, if one desires to pass from the space $\mathcal{P}_m$ to the space $\mathcal{P}_{m+1}$ by adjunction of $2m + 3$ points, one has to determine an entirely new system of fundamental polynomials that are not related in a simple fashion to the old set.

In the Newton representation, however, the increase of data can be accomplished simply by adding of additional terms as described now.

Suppose that $\{\eta_{M+Q}\}_{Q=1,2,\ldots}$ is a sequence of admissible systems of order m on Ω.

Then the (Gram) matrix

$$\sigma_M^{M+Q} = \begin{pmatrix} K_m^\circ(\eta_{M+1}, \eta_{M+1}) & \cdots & K_m^\circ(\eta_{M+1}, \eta_{M+Q}) \\ \vdots & & \vdots \\ K_m^\circ(\eta_{M+Q}, \eta_{M+1}) & \cdots & K_m^\circ(\eta_{M+Q}, \eta_{M+Q}) \end{pmatrix} \tag{13.7}$$

is non-singular for each integer $Q \geq 1$.

Corresponding to the sequence $\{\eta_{M+Q}\}_{Q=1,2,\ldots}$ we introduce the sequence $\{L_{M+Q}\}_{Q=1,2,\ldots}$ of evaluation functionals defined by

$$L_{M+Q} f = f(\eta_{M+Q}) \quad , \quad Q = 1, 2, \ldots \quad .$$

Then, according to the biorthonormality theorem (cf. Davis (1963), Theorem 2.6.1), there are determined uniquely two triangular systems of constants $b_{i,j}$, $g_{i,j}$ with $b_{ii} \neq 0$ such that if

$$
\begin{aligned}
L^{*}_{M+1} &= b_{1,1} L_{M+1} \\
L^{*}_{M+2} &= b_{2,1} L_{M+1} + b_{2,2} L_{M+2} \\
&\vdots
\end{aligned}
$$

and (13.8)

$$
\begin{aligned}
k^{*}_{M+1}(\zeta) &= K^{\circ}_{m}(\zeta_{M+1}, \zeta) \\
k^{*}_{M+2}(\zeta) &= g_{2,1} K^{\circ}_{m}(\zeta_{M+1}, \zeta) + K^{\circ}_{m}(\zeta_{M+2}, \zeta) \\
&\vdots
\end{aligned}
$$

we have

$$
L^{*}_{M+T}\, k^{*}_{M+Q} = k^{*}_{M+Q}(\zeta^{*}_{M+T}) = \delta_{QT}\,, \quad Q, T = 1, 2, 3, \ldots
$$

i.e.:

$$
L^{*}_{M+T}\, k^{*}_{M+Q}
$$

$$
= k^{*}_{M+Q}(\zeta^{*}_{M+T})
$$

$$
= \begin{cases} 1 & Q = T \\ 0 & Q \neq T \,. \end{cases}
$$

The result of biorthogonalization can be formally expressed by means of determinants

$$L^{\bullet}_{M+j} = \frac{1}{|\sigma_M^{M+j}|} \begin{vmatrix} K_m^\circ(\eta_{M+1},\eta_{M+1}) & \dots & K_m^\circ(\eta_{M+1},\eta_{M+j}) \\ \vdots & & \vdots \\ K_m^\circ(\eta_{M+j-1},\eta_{M+1}) & \dots & K_m^\circ(\eta_{M+j-1},\eta_{M+j}) \\ L_{M+1} & \dots & L_{M+j} \end{vmatrix}$$

$$k^{\bullet}_{M+j}(\eta) = \frac{1}{|\sigma_M^{M+j-1}|} \begin{vmatrix} K_m^\circ(\eta_{M+1},\eta_{M+1}) & \dots & K_m^\circ(\eta_{M+j},\eta_{M+1}) \\ \vdots & & \vdots \\ K_m^\circ(\eta_{M+1},\eta_{M+j-1}) & \dots & K_m^\circ(\eta_{M+j},\eta_{M+j-1}) \\ K_m^\circ(\eta_{M+1},\eta) & \dots & K_m^\circ(\eta_{M+j},\eta) \end{vmatrix} .$$

Of great interest is that for the computation of the last determinant only the points $\eta_{M+1}, \dots, \eta_{M+j}$ are needed from our prescribed admissible system of order m.

Consider a function f of the subspace spanned by the functions $K_m^\circ(\eta_{M+1},\eta), \dots, K_m^\circ(\eta_{M+Q},\eta)$, i.e. the set of all linear combinations of the functions $K_m^\circ(\eta_{M+1},\eta), \dots, K_m^\circ(\eta_{M+Q},\eta)$:

$$f \in \operatorname{span}\left(K_m^{\circ}(\eta_{M+1}, \eta), \ldots, K_m^{\circ}(\eta_{M+Q}, \eta)\right).$$

Then

$$f(\eta) = \sum_{n=1}^{Q} f(\eta_{M+n}^{*})\, k_{M+n}^{*}(\eta) . \qquad (13.9)$$

The representation (13.9) is called (generalized) Newton formula.

□

Remark: For the following considerations it will help to describe the biorthonormality theorem of Newton type in the language of matrices.

Let β and γ designate the triangular matrices taken from the coefficient scheme (13.8):

$$\beta = (b_{i,j})_{\substack{i=1,\ldots,Q \\ j=1,\ldots,Q}}$$

and

$$\gamma = (g_{i,j})_{\substack{i=1,\ldots,Q \\ j=1,\ldots,Q}} .$$

Then it is characteristic for the biorthonormality theorem (cf. Davis (1963), chapt. II, sect. 2.6) that

$$\beta\, \sigma_M^{M+Q}\, \gamma^{t} = I \qquad (13.10a)$$

(I: unit matrix). This is equivalent to

$$\sigma_M^{M+Q} = \beta^{-1} \gamma^{-t} . \qquad (13.10b)$$

Now, β^{-1} is a lower triangular matrix with non-zero elements on its principal diagonal and γ^{-t} is an upper triangular matrix. Since the matrix σ_M^{M+Q} is positive definite and symmetric, the decomposition (13.10b) is actually a $\mu^{t} \delta \mu$ - factorization, where μ is an upper triangular and δ is a positive diagonal matrix. Therefore, there exists a unique

upper triangular matrix τ_M^{M+Q} with positive diagonal elements such that

$$\sigma_M^{M+Q} = (\tau_M^{M+Q})^t \, \tau_M^{M+Q}.$$

This (uniquely defined) splitting of σ_M^{M+Q} is known as the Cholesky factorization (Cholesky decomposition).

□

Our purpose now is to combine the advantages of the Lagrange formula with the flexibility of the Newton formula. As a matter of fact, the solution process of optimal spherical spline interpolation can be achieved in a recursive form.

As shown in Chapt. 11, the problem of determining an (optimal) interpolating spline to the data

$$(\eta_1, y_1) \quad , \quad \ldots \quad , \quad (\eta_{M+Q}, y_{M+Q}) \tag{13.11}$$

is equivalent to the problem of finding the function $s \in \mathcal{S}_m(\eta_1, \ldots, \eta_{M+Q})$ of the form

$$s(\eta) = \sum_{n=1}^{M} y_n \mathcal{B}_n(\eta) + \sum_{k=M+1}^{M+Q} a_k K_m^{\circ}(\eta_k, \eta) \tag{13.12}$$

to the vector $a \in \mathbb{R}^Q$, $a^t = (a_{M+1}, \ldots, a_{M+Q})$ satisfying the linear equations

$$\sum_{k=M+1}^{M+Q} a_k \, K_m^\circ(\eta_k, \eta_j) = y_j - \sum_{n=1}^{M} y_n \, \mathcal{B}_n(\eta_j) \tag{13.13}$$

for $j = M+1, \ldots, M+Q$.

Using the notation (13.7) the linear equations can be rewritten in the vectorial form

$$G_M^{M+Q} \, a = \tilde{y}, \tag{13.14}$$

where the vector $\tilde{y} \in \mathbb{R}^Q$, $(\tilde{y})^t = (\tilde{y}_{M+1}, \ldots, \tilde{y}_{M+Q})$ is given by

$$\tilde{y}_j = y_j - \sum_{n=1}^{M} y_n \, \mathcal{B}_n(y_j) . \tag{13.15}$$

Furthermore, setting for arbitrary but fixed $\eta \in \Omega$,

$$\tilde{K}(\eta) \in \mathbb{R}^Q , (\tilde{K}(\eta))^t = (K_m^\circ(\eta_{M+1}, \eta), \ldots, K_m^\circ(\eta_{M+Q}, \eta)) \tag{13.16}$$

we are able to express the linear combination

$$\sum_{k=M+1}^{M+Q} a_k \, K_m^\circ(\eta_k, \eta) \tag{13.17}$$

as scalar product

$$a^t \, \tilde{K}(\gamma) \; . \tag{13.18}$$

The (Q,Q) - matrix σ_M^{M+Q} is positive definite and symmetric. Thus, according to Cholesky's factorization theorem, there exists a non-singular, upper triangular matrix τ_M^{M+Q} (with positive diagonal elements) such that

$$\sigma_M^{M+Q} = (\tau_M^{M+Q})^t \; \tau_M^{M+Q} \tag{13.19}$$

($(\quad)^t$: transposed matrix). But this yields in connection with (13.14)

$$a = (\sigma_M^{M+Q})^{-1} \, \tilde{y}$$

$$= (\tau_M^{M+Q})^{-1} \, (\tau_M^{M+Q})^{-t} \, \tilde{y} \; . \tag{13.20}$$

Thus our expression (13.17) takes the form

$$\tilde{y}^t \, (\tau_M^{M+Q})^{-1} (\tau_M^{M+Q})^{-t} \, \tilde{K}(\gamma) . \tag{13.21}$$

Using the abbreviations

$$y^* \in \mathbb{R}^Q \,, \quad y^* = (\tau_M^{M+Q})^{-t}\, \tilde{y} \tag{13.22}$$

and, for each $\eta \in \Omega$,

$$k^*(\eta) \in \mathbb{R}^Q \,, \; k^*(\eta) = (\tau_M^{M+Q})^{-t}\, \tilde{k}(\eta) \tag{13.23}$$

we get

$$\tilde{y}^t\, (\tau_M^{M+Q})^{-1}\, (\tau_M^{M+Q})^{-t}\, \tilde{k}(\eta)$$

$$= \; (y^*)^t \; k^*(\eta) \,.$$

Using coordinates

$$y^* \in \mathbb{R}^Q \,, \; (y^*)^t = (y^*_{M+1}, \ldots, y^*_{M+Q})$$

$$k^*(\eta) \in \mathbb{R}^Q \,, \; (k^*(\eta))^t = (k^*_{M+1}(\eta), \ldots, k^*_{M+Q}(\eta))$$

this can be expressed as follows

$$\sum_{k=M+1}^{M+Q} a_k \; k_m^\circ(\eta_k, \eta)$$

$$= \sum_{n=1}^{Q} y^*_{M+n} \; k^*_{M+n}(\eta) \,.$$

The sum of the right hand side of the last formula represents a formula of Newton type (for algorithmic details cf. e.g. Davis (1963), Meinguet (1979)). Furthermore, the computation of y^*, $k^*(\eta)$ can be organized, indeed, easily by forward substitution.

Summarizing our results we therefore obtain the following result:

Let $\eta_1, \ldots, \eta_M$ be a fundamental system of order m on the unit sphere Ω. Then each spline function $s \in \mathcal{S}_m(\eta_1, \ldots, \eta_{M+Q})$ of the form (13.12) can be represented in the form

$$s(\eta) = \sum_{n=1}^{M} \gamma_n B_n(\eta) + \sum_{j=1}^{Q} \gamma^*_{M+j} k^*_{M+j}(\eta), \tag{13.24}$$

where the vector $y^* \in \mathbb{R}^Q$, $(y^*)^t = (y^*_{M+1}, \ldots, y^*_{M+Q})$ is the solution of the linear system

$$(\tau_M^{M+Q})^t \, y^* = \tilde{y} \tag{13.25}$$

and, for each but fixed $\eta \in \Omega$, $k^*(\eta) \in \mathbb{R}^Q$, $(k^*(\eta))^t = (k^*_{M+1}(\eta), \ldots, k^*_{M+Q}(\eta))$ is the solution of the linear system

$$(\tau_M^{M+Q})^t \, k^*(\eta) = \tilde{k}(\eta). \tag{13.26}$$

Herein τ_M^{M+Q}, $(\tau_M^{M+Q})^t$ are the uniquely determined Cholesky - factors of σ_M^{M+Q}, i.e.

$$\sigma_M^{M+Q} = (\tau_M^{M+Q})^t \tau_M^{M+Q}, \quad \tau_M^{M+Q} \text{ is an upper triangular.}$$

The first sum of the right hand side of (13.24) is a sum of Lagrange type, while the second sum is of Newton type.

□

Remark: The Cholesky factorization is one of the best direct methods for solving linear systems with a positive definite, symmetric matrix. The computer implementation of Cholesky's decomposition is simple. Economy of storage can be achieved by working only with a linear array of $Q(Q+1)/2$ elements consisting initially of the upper triangle of σ_M^{M+Q} which is later overwritten with τ_M^{M+Q}. This method is also economical as regards the number of arithmetic operations. As is well-known, at most Q square roots and approximately $Q^3/6$ operations (1 operation = 1 multiplication + 1 addition) are required, to be compared with about $Q^3/3$ operations for the well-known scheme of Banachiewicz.

For the computational procedures, well-known routines (in double precision, if necessary) are available (cf. e.g. Wilkinson - Reinsch (1971)).

□

The spline of the form (13.24) possesses the permanence property, i.e. the transition from the spline $s^{(M+Q)} \in \mathcal{S}_m(\tau_1, \cdots, \tau_{M+Q})$ interpolating the data points (τ_1, y_1), $\cdots$, (τ_{M+Q}, y_{M+Q}) to the spline $s^{(M+Q+1)} \in \mathcal{S}_m(\tau_1, \cdots, \tau_{M+Q+1})$ interpolating the data points $(\tau_1, y_1), \cdots, (\tau_{M+Q+1}, y_{M+Q+1})$ necessitates merely the addition of one more term to the expansion of $s^{(M+Q)}$, all the terms obtained formerly remaining unchanged:

$$s^{(M+Q+1)}(\tau) = \sum_{n=1}^{M} y_n B_n(\tau) + \sum_{j=1}^{Q+1} y^*_{M+j} k^*_{M+j}(\tau)$$

$$= s^{(M+Q)}(\tau) + y^*_{M+Q+1} k^*_{M+Q+1}(\tau).$$

The price to be paid is the change of the basis system for the spherical harmonics of order $\leq m$ and the biorthonormalization process.

The reasons for the convenience of the permanence property are the following:

(i) by virtue of the Cholesky - decomposition of the matrix $\sigma_M^{M+Q} = (\tau_M^{M+Q})^t \tau_M^{M+Q}$ the i-th row of τ_M^{M+Q} depends only on the rows $j \leq i$ of the matrix σ_M^{M+Q}.

(ii) the quantities y^*, $k^*(\tau)$ are computed by forward substitution.

From an algorithmic point of view, the transition from the spline

$$s^{(M+Q)}(\eta) = \sum_{n=1}^{M} y_n B_n(\eta) + \sum_{j=1}^{Q} y^*_{M+j} k^*_{M+j}(\eta)$$

interpolating the data points

$$(\eta_1, y_1), \quad \ldots, \quad (\eta_{M+Q}, y_{M+Q})$$

to the spline

$$s^{(M+Q+T)}(\eta) = \sum_{n=1}^{M} y_n B_n(\eta) + \sum_{j=1}^{Q+T} y^*_{M+j} k^*_{M+j}(\eta)$$

interpolating the data points

$$(\eta_1, y_1), \quad \ldots, \quad (\eta_{M+Q+T}, y_{M+Q+T})$$

(T: positive integer) necessitates merely the computation of the additional terms in σ_M^{M+Q+T} and the continuation of the Cholesky factorization from σ_M^{M+Q} to σ_M^{M+Q+T}.

According to the biorthonormalization process, exclusively forward substitution (cf. (13.25), (13.26)) is required for the calculation of the sum

$$\sum_{j=Q+1}^{Q+T} y^*_{M+j} k^*_{M+j}(\eta) .$$

Remark: Generalizations to heterogeneous data (in form of (bounded) linear functionals) will be given in Reuter (1982) and Freeden/Reuter (1982).

Analogous considerations establish the permanence property in best approximation problems. □

Remark:

In modified form the interpolation method analyzed here has been proposed first by Meinguet (1979) in connection with the Beppo - Levi space $\mathcal{H}^{(m)}(\mathbb{R}^q)$ of continuous linear functionals defined on the class of infinitely differentiable functions with compact support in q - dimensional Euclidean space $\mathbb{R}^q$, for which all the partial derivatives

$$\frac{\partial^{[\alpha]}}{\partial x_1^{\alpha_1} \dots \partial x_q^{\alpha_q}} f, \qquad [\alpha] = \alpha_1 + \dots + \alpha_q,$$

(in the distributional sense) of total order m are square - integrable in $\mathbb{R}^q$. In this case, $\mathcal{H}^{(m)}(\mathbb{R}^q)$ is naturally equipped with the semi-inner product corresponding to the norm

$$\left\{ \sum_{\alpha_1, \dots, \alpha_q = m} \int_{\mathbb{R}^q} \left| \frac{\partial^{[\alpha]}}{\partial x_1^{\alpha_1} \dots \partial x_q^{\alpha_q}} f(x) \right|^2 d(V'x) \right\}^{1/2},$$

(dV: volume element)

where every partial derivative is to be interpreted in the distributional sense. □

14. COMPUTATIONAL ASPECTS

The results developed in this paper give rise to the following algorithm:

Step 1: compute the symmetric matrix (N = M + Q)

$$\begin{pmatrix} \mathcal{G}^{(2)}(\lambda_0,\dots,\lambda_m;\gamma_1,\gamma_1) & \cdots & \mathcal{G}^{(2)}(\lambda_0,\dots,\lambda_m;\gamma_1,\gamma_N) \\ \vdots & & \vdots \\ \mathcal{G}^{(2)}(\lambda_0,\dots,\lambda_m;\gamma_N,\gamma_1) & \cdots & \mathcal{G}^{(2)}(\lambda_0,\dots,\lambda_m;\gamma_N,\gamma_N) \end{pmatrix}$$

Step 2: print the matrix

$$\begin{pmatrix} S_{0,1}(\gamma_1) & \cdots & S_{0,1}(\gamma_N) \\ S_{1,1}(\gamma_1) & \cdots & S_{1,1}(\gamma_N) \\ \vdots & & \vdots \\ S_{1,3}(\gamma_1) & \cdots & S_{1,3}(\gamma_N) \\ \vdots & & \vdots \\ S_{m,1}(\gamma_1) & \cdots & S_{m,1}(\gamma_N) \\ \vdots & & \vdots \\ S_{m,2m+1}(\gamma_1) & \cdots & S_{m,2m+1}(\gamma_N) \end{pmatrix}$$

Step 3: compute the matrix

$$\begin{pmatrix} \mathcal{B}_1(\eta_{M+1}) & \dots & \mathcal{B}_1(\eta_N) \\ \vdots & & \vdots \\ \mathcal{B}_M(\eta_{M+1}) & \dots & \mathcal{B}_M(\eta_N) \end{pmatrix}$$

$$= \begin{pmatrix} S_{0,1}(\eta_1) & \dots & S_{0,1}(\eta_M) \\ \vdots & & \vdots \\ S_{m,2m+1}(\eta_1) & \dots & S_{m,2m+1}(\eta_M) \end{pmatrix}^{-1} \begin{pmatrix} S_{0,1}(\eta_{M+1}) & \dots & S_{0,1}(\eta_N) \\ \vdots & & \vdots \\ S_{m,2m+1}(\eta_{M+1}) & \dots & S_{m,2m+1}(\eta_N) \end{pmatrix}$$

by solving the linear system

$$\begin{pmatrix} S_{0,1}(\eta_1) & \dots & S_{0,1}(\eta_M) \\ \vdots & & \vdots \\ S_{m,2m+1}(\eta_1) & \dots & S_{m,2m+1}(\eta_M) \end{pmatrix} \begin{pmatrix} \mathcal{B}_1(\eta_{M+1}) & \dots & \mathcal{B}_1(\eta_N) \\ \vdots & & \vdots \\ \mathcal{B}_M(\eta_{M+1}) & \dots & \mathcal{B}_M(\eta_N) \end{pmatrix}$$

$$= \begin{pmatrix} S_{0,1}(\eta_{M+1}) & \dots & S_{0,1}(\eta_N) \\ \vdots & & \vdots \\ S_{m,2m+1}(\eta_{M+1}) & \dots & S_{m,2m+1}(\eta_N) \end{pmatrix}$$

Step 4: compute the symmetric matrix

$$\sigma_M^N = \begin{pmatrix} K_m^\circ(\eta_{M+1}, \eta_{M+1}) & \dots & K_m^\circ(\eta_{M+1}, \eta_N) \\ \vdots & & \vdots \\ K_m^\circ(\eta_N, \eta_{M+1}) & \dots & K_m^\circ(\eta_N, \eta_N) \end{pmatrix},$$

where $K_m^\circ(\eta_i, \eta_j)$ is given by (10.14)

$$K_m^\circ(\eta_i,\eta_j) = \mathcal{G}^{(2)}(\lambda_0,\dots,\lambda_m;\eta_i,\eta_j)$$

$$- \sum_{k=1}^{M} \mathcal{G}^{(2)}(\lambda_0,\dots,\lambda_m;\eta_i,\eta_k)\,\mathcal{B}_k(\eta_j)$$

$$- \sum_{k=1}^{M} \mathcal{B}_k(\eta_i)\,\mathcal{G}^{(2)}(\lambda_0,\dots,\lambda_m;\eta_k,\eta_j)$$

$$+ \sum_{k=1}^{M}\sum_{n=1}^{M} \mathcal{B}_k(\eta_i)\,\mathcal{G}^{(2)}(\lambda_0,\dots,\lambda_m;\eta_k,\eta_n)\,\mathcal{B}(\eta_j)$$

Step 5: Cholesky - factorization of the matrix $\left(K_m^\circ(\eta_i,\eta_j)\right)_{\substack{i=M+1,\dots,N\\ j=M+1,\dots,N}}$

$$\sigma_M^N = (\tau_M^N)^t \quad \tau_M^N$$

Step 6: determine the vector $\tilde{y}$ given by

$$\tilde{y}_{M+1} = y_{M+1} - \sum_{n=1}^{M} y_n\,\mathcal{B}_n(\eta_{M+1})$$

$$\vdots \qquad\qquad \vdots$$

$$\tilde{y}_N = y_N - \sum_{n=1}^{M} y_n\,\mathcal{B}_n(y_N)$$

Step 7: solve the linear system (forward substitution)

$$(\tau_M^N)^t\, y^* = \tilde{y}$$

Step 8: choose the point $\eta \in \Omega$, for which the interpolating spline has to be computed

Step 9: compute the vector $\mathcal{B}(\eta)$, $(\mathcal{B}(\eta))^t = (\mathcal{B}_1(\eta), \ldots, \mathcal{B}_M(\eta))$, by solving the linear system

$$\begin{pmatrix} S_{0,1}(\eta_1) & \ldots & S_{0,1}(\eta_M) \\ \vdots & & \vdots \\ S_{m,2m+1}(\eta_1) & \ldots & S_{m,2m+1}(\eta_M) \end{pmatrix} \begin{pmatrix} \mathcal{B}_1(\eta) \\ \vdots \\ \mathcal{B}_M(\eta) \end{pmatrix} = \begin{pmatrix} S_{0,1}(\eta) \\ \vdots \\ S_{m,2m+1}(\eta) \end{pmatrix}$$

Step 10: compute the vector $\widetilde{K}(\eta)$ given by

$$(\widetilde{K}(\eta))^t = (K_m^\circ(\eta_{M+1}, \eta), \ldots, K_m^\circ(\eta_N, \eta))$$

Step 11: solve the linear system (forward substitution)

$$(\tau_M^N)^t k^*(\eta) = \widetilde{K}(\eta)$$

Step 12: compute the interpolating spline s of the form (13.24) at the point $\eta \in \Omega$

Step 13: continue with Step 8 □

REMARK: The matrix

$$\begin{pmatrix} S_{0,1}(\eta_1) & \ldots & S_{0,1}(\eta_M) \\ \vdots & & \vdots \\ S_{m,2m+1}(\eta_1) & \ldots & S_{m,2m+1}(\eta_M) \end{pmatrix}^{-1}$$

is not used anywhere in our algorithm.

15. A SIMPLE EXAMPLE

We consider the potential of the(buried)mass point

$$u(\eta) = \frac{1}{|\eta - y|}, \quad \eta \in \Omega, \quad y^t = (0, 0, 0.9). \tag{15.1}$$

Our aim is to approximate u by a spherical spline function s_u interpolating in given points $\eta_1, \ldots, \eta_N$ of the unit sphere. We chose 62 points generated by regular polyhedra

$$\mathcal{T}_1: \begin{array}{l} \beta(0,\pm\tau,\pm 1),\ \beta(\pm 1,0,\pm\tau),\ \beta(\pm\tau,\pm 1,0), \\ \gamma(\pm\beta,\pm\tau,0),\ \gamma(0,\pm\beta,\pm\tau),\ \gamma(\pm\tau,0,\pm\beta),\ \gamma(\pm 1,\pm 1,\pm 1), \\ (\pm 1,0,0),\ (0,\pm 1,0),\ (0,0,\pm 1), \end{array}$$

where τ, β, γ are given by

$$\tau = (1 \pm \sqrt{5})/2, \quad \beta = 1/\sqrt{1+\tau^2}, \quad \gamma = 1/\sqrt{3}$$

We assume polynomial precision up to m = 4.

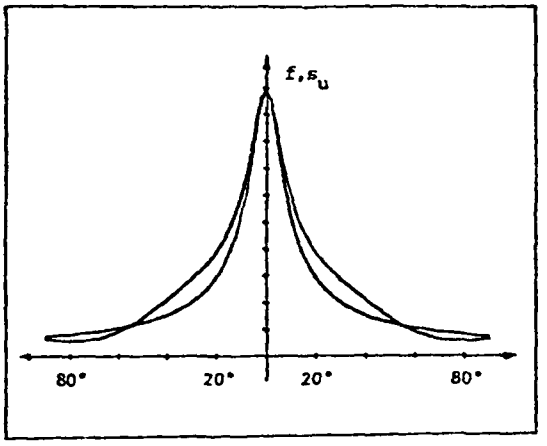

Figure 1 gives a graphical impression of the accuracy for a meridian on the northern hemisphere passing through the north pole.

In the following we have used two times eight additional nodes to eliminate the derivatives between the interpolating spline and the function u in a local area around the north pole.

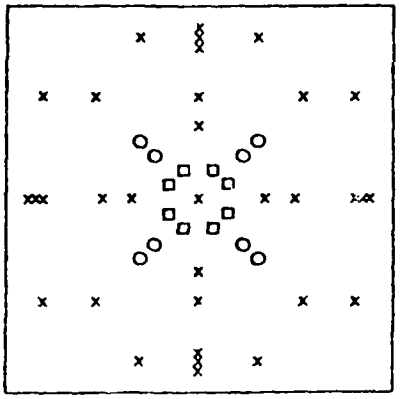

Figure 2 illustrates the distribution of the points of the northern hemisphere (projected into the equatorial plane).
"x" denotes the points of $\mathcal{T}_1$, while "o" and "□" stands for the nodes of $\mathcal{T}_2$ and $\mathcal{T}_3$ respectively.

$\mathcal{T}_2$: (±0.2500, ±0.2500, 0.9354)
(±0.3536, ±0.3536, 0.8660)

$\mathcal{T}_3$: (±0.1721, ±0.0713, 0.9825)
(±0.0713, ±0.1721, 0.9825)

For comparison Figure 3 shows the graph of the interpolating spline with respect to the 70 points of $\mathcal{T}_1 \cup \mathcal{T}_2$, while Figure 4 is based on all 78 points.

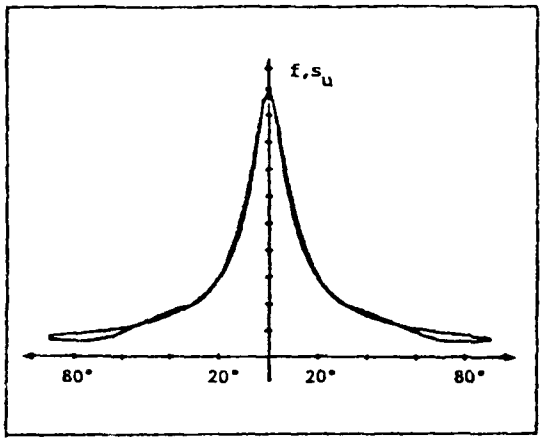

Figure 3

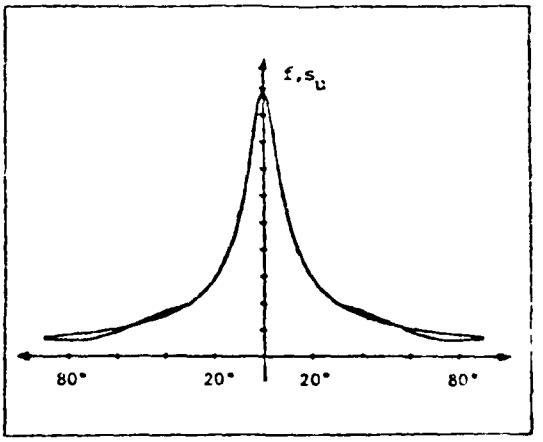

Figure 4

It turns out that we get an acceptable approximation in the neighborhood of the north pole avoiding a deterioration of the accuracy in all other parts of the northern hemisphere.

Unfortunately it appears that the Green function $g^{(2)}(\lambda_0,\ldots,\lambda_m;\xi,\eta)$ with respect to the operator $(\Delta^*)^2_m$ cannot be expressed by an elementary function. Thus the Green function has to be replaced by an appropriate expression which is convenient for computational purposes.

In our example, satisfactory results can be achieved based on finite partial sums of the bilinear expansion (6.11).

Other suggestions for replacing Green's (kernel) functions by closed forms convenient for computation will be discussed in Reuter (1982) and Freeden/Reuter (1982).

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